Groupoid Atlases

A. Bak, R. Brown, G. Minian and T. Porter

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Abstract

Global actions were introduced by the first author to give a purely algebraic / combinatorial approach to algebraic K-theory. Using the fact that group actions yield action groupoids, we present here a variant of global actions that we call *groupoid atlases*. These consist of an interrelated family of 'locally defined' groupoids on a set. We discuss some motivating examples and explore some of the elementary theory.

1 Global actions

The motivation for the introduction of global actions by A. Bak was in part to provide an algebraic setting in which to perform the homotopic operations needed in algebraic K-theory without the heavy weight algebraic topology customarily used.

The motivating idea is of a family of interacting and overlapping local G-sets for varying G. The prime example is the underlying set $Gl_n(R)$ operated on by the family of subgroups $Gl_n(R)_\alpha$ generated by elementary matrices of a certain form. We will give the details shortly. First the definition of a global action.

A **global action** A consists of a set X_A together with a family

$$\{(G_A)_\alpha \curvearrowright (X_A)_\alpha \mid \alpha \in \Phi_A\}$$

of group actions on subsets $(X_A)_\alpha \subseteq X_A$. The various **local groups** $(G_A)_\alpha$ and the corresponding subsets $(X_A)_\alpha$ are indexed by the index set Φ_A, called the **coordinate system** of A. This set Φ_A is equipped with a reflexive relation, written $\leq$, and it is required that

- if $\alpha \leq \beta$ in Φ_A, then $(G_A)_\alpha$ leaves $(X_A)_\alpha \cap (X_A)_\beta$ invariant,

and

- there is given for each pair $\alpha \leq \beta$, a group homomorphism

$$(G_A)_{\alpha\leq\beta} : (G_A)_\alpha \to (G_A)_\beta$$

such that if $\sigma \in (G_A)_\alpha$ and $x \in (X_A)_\alpha \cap (X_A)_\beta$ then

$$\sigma x = (G_A)_{\alpha\leq\beta}(\sigma)x.$$

The diagram $G_A : \Phi_A \to$ *Groups* is called the **global group** of A. The set X_A is the **enveloping set** or **underlying set** of A. The notation $|X_A|$ or $|A|$ for X_A is sometimes used for emphasis or to avoid confusion since

$$X_A : \Phi_A \to \mathcal{P}(X_A)$$

is also a useful notation, where $\mathcal{P}(X_A)$ is the powerset of X_A.

REMARKS

a) For technical reasons it is not assumed that the collection $(X_A)_\alpha \subseteq X_A$ necessarily covers X_A. This is so in all the basic examples we will examine but is not a requirement.

b) The relation $\leq$ is not assumed to be transitive on Φ_A, so really G_A is not a functor, however the difference is minor as, if $F(\Phi_A)$ denotes the free category on the graph of $(\Phi_A, \leq)$, then G_A extends to a functor $G_A : F(\Phi_A) \to$ *Groups*. We will usually refer, as here, to G_A as a diagram of groups and will sometimes use 'natural transformation' to mean a generalised natural transformation defined on the generating graphs, that yields an actual natural transformation on the corresponding extensions.

The simplest global actions come with just a single domain.

A global action A is said to be **single domain** if for each $\alpha \in \Phi_A, (X_A)_\alpha = |A|$.

EXAMPLE

Let G be a group, $\mathcal{H} = \{H_i : i \in \Phi\}$ a family of subgroups of G. For the moment Φ is just a set (that is : $\alpha \leq \beta$ in Φ if and only if $\alpha = \beta$). Define $A = A(G, \mathcal{H})$ to be the global action with

$$X = |X_A| = |G|, \text{ the underlying set of } G$$
$$\Phi_A = \Phi$$
$$(X_A)_\alpha = X_A \text{ for all } \alpha \in \Phi$$
$$H_i \curvearrowright X \text{ by left multiplication}$$

(so the local orbits of the H_i-action are the left cosets of H_i).

WARNING

Later on we will need to refine this construction, taking Φ_A to be the family of finite non-empty subsets of Φ ordered by opposite inclusion and with if $\alpha \in \Phi_A$, $(G_A)_\alpha = \bigcap_{i \in \alpha} H_i$.

We will later look in some detail at certain specific such single domain global actions. The following prime motivating example is similar to these, but the indexing set/coordinate system is slightly more complex.

The General Linear Global Action $\boldsymbol{Gl}_n(R)$

R will be an associative ring with identity and n a positive integer.

Let $\Delta = \{(i,j) \mid i \neq j, 1 \leq i,j \leq n\}$ be the set of non-diagonal positions in a $n \times n$ array. Call a subset $\alpha \subseteq \Delta$ *closed* if

$$(i,j) \in \alpha \text{ and } (j,k) \in \alpha \text{ implies } (i,k) \in \alpha$$

Note if $(i,j) \in \alpha$ and α is closed then $(j,i) \notin \alpha$.

Let $\Phi = \{\alpha \subseteq \Delta : \alpha \text{ is closed}\}$. We put a reflexive relation $\leq$ on Φ by $\alpha \leq \beta$ if $\alpha \subseteq \beta$.

Now suppose $(i,j) \in \Delta$ and $r \in R$. The elementary matrix $\epsilon_{ij}(r)$ is the matrix obtained from the identity $n \times n$ matrix by putting the element r in position (i,j),

$$\text{i.e.} \qquad \epsilon_{ij}(r)_{k,l} = \begin{cases} 1 & \text{if } k = l \\ r & \text{if } (k,l) = (i,j) \\ 0 & \text{otherwise .} \end{cases}$$

Let $Gl_n(R)_\alpha$, for $\alpha \in \Phi$, denote the subgroup of $Gl_n(R)$ generated by

$$\{\epsilon_{ij}(r) \mid (i,j) \in \alpha, r \in R\}.$$

It is easy to see that $(a_{kl}) \in Gl_n(R)_\alpha$ if and only if

$$a_{k,l} = \begin{cases} 1 & \text{if } k = l \\ \text{arbitrary} & \text{if } (i,j) \in \alpha \\ 0 & \text{if } (i,j) \in \Delta \backslash \alpha. \end{cases}$$

For $\alpha \leq \beta$ there is an inclusion of $Gl_n(R)_\alpha$ into $Gl_n(R)_\beta$. This will give the homomorphism

$$Gl_n(R)_{\alpha \leq \beta} : Gl_n(R)_\alpha \to Gl_n(R)_\beta.$$

Let $Gl_n(R)_\alpha$ act by left multiplication on $Gl_n(R)$.

This completes the description of the single domain global action $\mathbf{Gl}_n(R)$. Later we will see how to define the homotopy groups of a global action. The $(i-1)^{\text{th}}$-homotopy group of $\mathbf{Gl}_n(R)$ is the algebraic K-theory group $K_i(n,R)$ and the usual algebraic K-group, $K_i(R)$ is the direct limit of $K_i(n,R)s$ by the obvious maps induced from the inclusions $\mathbf{Gl}_n(R) \to \mathbf{Gl}_{n+1}(R)$.

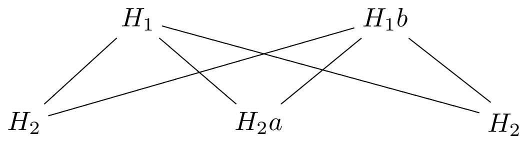
The way that a global action extends local information to become global information can be observed from the simplest cases of the $A(G,\mathcal{H})$.

If $\mathcal{H}$ has just a single group H in it, then the global action is just the collection of orbits, i.e. right cosets. There is no interaction between them.

If $\mathcal{H} = \{H_1, H_2\}$, then any H_1-orbit intersects with any H_2-orbit, so now orbits do interact. How they interact can be very influential on the homotopy properties of the situation. As an example consider the symmetric group

$$S_3 \equiv \langle a, b \mid a^3 = b^2 = (ab)^2 = 1\rangle,$$

with a denoting the 3-cycle $(1\ 2\ 3)$ and b the transposition $(1\ \ 2)$. Take $H_1 = \langle a\rangle = \{1, (1\ 2\ 3), (1\ 3\ 2)\}$ yielding two orbits for its left action on S_3, H_1 and $H_1 b$. Similarly take $H_2 = \langle b\rangle$ giving local orbits $H_2, H_2 a, H_2 a^2$. Any H_1-orbit intersects with any H_2-orbit, but of course they do not overlap themselves. This gives an intersection diagram:



This graph makes it clear that, even in such a simple case, it is possible to find loops and circuits within the global action, following an element through a local orbit and then, within an intersection, crossing to the next orbit, eventually getting back to the starting position.

The element $1 \in H_2$ multiplying again on the left by $b \in H_2$ ends up in $H_2 \cap H_1 b$, multiplied on the left by $a \in H_1$ yields $ab \in H_1 b \cap H_2 a^2$ and so on. The circuit

$$H_2 \longrightarrow H_1 b \longrightarrow H_2 a^2 \longrightarrow H_1 \longrightarrow H_2$$

$$I \xrightarrow{b\times} b \xrightarrow{a\times} ab \xrightarrow{b\times} bab \xrightarrow{a\times} abab = 1$$

relates the structure of the single domain global action with the combinatorial information encoded in the presentation. This will be examined in more detail later.

Morphisms

Morphisms between global actions come in various strengths depending on what part of the data is preserved. Preservation of just the local orbit information corresponds to a "morphism", compatibility with the whole of the data then yields a "regular morphism".

First we introduce a subsidiary notion which will be important at several points in the later development.

Definition

Let A be a global action. Let $x \in (X_A)_\alpha$ be some point in a local set of A.

A *local frame* at x in α or *α-frame* at x is a sequence $x = x_0, \cdots, x_p$ of points in the local orbit of the $(G_A)_\alpha$-action on $(X_A)_\alpha$ determined by x. Thus for each i, $1 \leq i \leq p$, there is some $g_i \in (G_A)_\alpha$ with $g_i x = x_i$.

Note that in extreme cases, such as a trivial action, all the x_i may be equal, but if the action is faithful, each α-frame at x essentially consists of x and a sequence $g_1, \cdots, g_p$ of elements of $(G_A)_\alpha$. For some of the homotopy theoretic side of the development this may be of use as $g_1, g_2 g_1^{-1}, \cdots$ yields a $(p-1)$-simplex in the nerve of the group $(G_A)_\alpha$.

Definition

If A and B are global actions, a **morphism** $f : A \to B$ of global actions is a function $f : |A| \to |B|$ on their underlying sets, which preserves local frames. More precisely:
if $x_0, \cdots, x_p$ is an α-frame at x_0 for some $\alpha \in \Phi_A$, then $f(x_0), \cdots, f(x_p)$ is a β-frame at $f(x_0)$ for some $\beta \in \Phi_B$.

Note that not all α-frames may lead to the same β, so this notion is **not** saying that the whole of the local orbit of the $(G_A)_\alpha$-action corresponding to x_0 must end up within a single local orbit, merely that given $x_0, \cdots, x_p$, there is some β such that $f(x_0), \cdots, f(x_p)$ form a β-frame. This is of course only significant when there are infinitely many co-ordinates, as larger frames may lead to different "larger" βs.

Intuitively a *path* in a global action A is a sequence of points $a_0, \cdots, a_n$ in $|A|$ so that each $a_i, a_{i+1}, i = 0, \cdots n-1$ is a α-frame for some (varying) $\alpha \in \Phi_A$. This idea can be captured using a morphism from a global action model of a line, and this is done in the initial papers on global actions, [1, 2]. Here we postpone this until the third section as there is a certain technical advantage in considering the line with a groupoid atlas structure and that will be introduced there.

Definition

A **regular morphism** $\eta : A \to B$ of global actions is a triple $(\eta_\Phi, \eta_G, \eta_X)$ satisfying the following

$$\begin{aligned} &-\eta_\Phi : \Phi_A \to \Phi_B \text{ is a relation preserving function :} \\ &\qquad \text{if } \alpha \leq \alpha', \text{ then } \eta_\Phi(\alpha) \leq \eta_\Phi(\alpha'), \\ &-\eta_G : G_A \to (G_A)_{\eta_\Phi} \text{ is a natural transformation of group diagrams over } \eta_\Phi, \end{aligned}$$

i.e. for each $\alpha \in \Phi_A$,

$$\eta_G(\alpha) : (G_A)_\alpha \to (G_B)_{\eta_\Phi(\alpha)}$$

is a group homomorphism such that if $\alpha \leq \alpha'$ in Φ_A, the diagram

$$\begin{array}{ccc} (G_A)_\alpha & \xrightarrow{\eta_G(\alpha)} & (G_B)_{\eta_\Phi(\alpha)} \\ \downarrow & & \downarrow \\ (G_A)_{\alpha'} & \xrightarrow[\eta_G(\alpha')]{} & (G_B)_{\eta_\Phi(\alpha')} \end{array}$$

commutes, where the vertical maps are the structure maps of the respective diagrams;

- $\eta_X : |A| \to |B|$ is a function such that $\eta_X((X_A)_\alpha) \subseteq (X_B)_{\eta_\Phi(\alpha)}$ for all $\alpha \in \Phi_A$;
- for each $\alpha \in \Phi_A$, the pair

$$(\eta_G, \eta_X) : (G_A)_\alpha \curvearrowright (X_A)_\alpha \to (G_B)_{\eta_\Phi(\alpha)} \curvearrowright (X_B)_{\eta_\Phi(\alpha)}$$

is a morphism of group actions.

Remarks:

If η is a regular morphism, it is clear that η_X preserves local frames and so is a morphism in the weaker sense.

Composition of both types of morphism is defined in the obvious way and so one obtains categories of global actions and morphisms and of global actions and regular morphisms.

It is perhaps necessary to underline the meaning of a morphism of group actions: if $G \curvearrowright X$ and $H \curvearrowright Y$ are group actions of G on X and H on Y, respectively, a morphism from $G \curvearrowright X$ to $H \curvearrowright Y$ is a pair $(\varphi : G \to H, \psi : X \to Y)$ with φ a homomorphism and ψ a function such that for $g \in G, x \in X$,

$$\varphi(g).\psi(x) = \psi(g.x).$$

We need to note that, if x and x' are in the same orbit of $G \curvearrowright X$ then $\psi(x)$ and $\psi(x')$ are in the same orbit of $H \curvearrowright Y$.

2 Actions as groupoids and groupoid atlases.

If $G \curvearrowright X$ is a group action then we can construct an action groupoid from it. (By a groupoid we mean a small category in which every arrow is an isomorphism.)

Act(G, X) or $G \ltimes X$ will denote the category with X as its set of objects and $G \times X$ as its set of arrows. Given an arrow (g, x), its source is x and its target $g.x$. We write $s(g, x) = x, t(g, x) = g.x$ and represent this diagrammatically by

$$x \xrightarrow{(g,x)} g.x \ .$$

The composite of (g, x) and (g', x') is defined only if the target of (g, x) is the source of (g', x') so $x' = g.x$, then

$$x \xrightarrow{(g,x)} g.x \xrightarrow{(g',gx)} g'gx$$

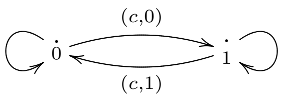
gives a composite $(g'g, x)$. The identity at x is $(1, x)$. The inverse of (g, x) is (g^{-1}, gx), so $G \ltimes X$ is a groupoid.

EXAMPLE

Let $X = \{0, 1\}, G = C_2$ with the obvious action on X interchanging 0 and 1. If we write $C_2 = \{1, c\}$, we have $\mathrm{Obj}(G \ltimes X) = X = \{0, 1\}$,

$$\mathrm{Arr}(G \ltimes X) = \{(1, 0) : 0 \to 0, (1, 1) : 1 \to 1, \quad (c, 0) : 0 \to 1, (c, 1) : 1 \to 0\}$$

Thus diagrammatically the groupoid is just

$G \ltimes X :=$ 

i.e. it is the groupoid often written as $\mathcal{I}$, the unit interval.

Back to the general situation:

Suppose $(\varphi, \psi) : G \curvearrowright X \to H \curvearrowright Y$ is a morphism of group actions, then define a morphism of groupoids by

$$\begin{array}{lll} \varphi \ltimes \psi : G \ltimes X & \to & H \ltimes Y \\ (\varphi \ltimes \psi)(x) & = & \psi(x) \qquad \text{on objects} \\ (\varphi \ltimes \psi)(g, x) & = & (\varphi(g), \psi(x)) \quad \text{on arrows.} \end{array}$$

We check:

$$\begin{aligned} s(\varphi(g), \psi(x)) &= \psi(x) = \psi(s(g, x)), \\ t(\varphi(g), \psi(x)) &= \varphi(g).\psi(x) = \psi(g, x) \\ &= \psi t(g, x). \end{aligned}$$

so $\varphi \ltimes \psi$ preserves source and target. It also preserves identities and composition as is easily checked.

The "language" of group actions thus translates well into the language of groupoids. The notion of an orbit of a group action becomes a connected component of a groupoid, so what is the analogue of a global action? The translation is not difficult, but the obvious term "global groupoid" does not seem to give the right intuition about the concept, so instead we will use the term "groupoid atlas".

First a bit of notation: if G is a groupoid with object set X and $X' \subset X$ is a subset of X then $G\lfloor_{X'}$ will denote the groupoid with object set X' having

$$G\lfloor_{X'} (x, y) = G(x, y),$$

if $x, y \in X'$ and with the same composition and identities as G, when this makes sense. This groupoid $G\lfloor_{X'}$ is the **full sub-groupoid of** G **determined by the objects in** X' or more simply, the **restriction** of G to X'.

DEFINITION

A **groupoid** atlas A on a set X_A consists of a family of groupoids $(\mathcal{G}_A)_\alpha$ defined with object sets $(X_A)_\alpha$, which are subsets of X_A. These **local groupoids** are indexed by an index set Φ_A, called the **coordinate system** of A, which is equipped with a reflexive relation, written $\leq$. This data is required to satisfy:

(i) if $\alpha \leq \beta$ in Φ_A, then $(X_A)_\alpha \cap (X_A)_\beta$ is a union of components of $(\mathcal{G}_A)_\alpha$,
i.e. if $x \in (X_A)_\alpha \cap (X_A)_\beta$ and $g \in (\mathcal{G}_A)_\alpha$ is such that $s(g) = x$, then $t(g) \in (X_A)_\alpha \cap (X_A)_\beta$;

and

(ii) if $\alpha \leq \beta$ in Φ_A, there is given a groupoid morphism

$$(\mathcal{G}_A)_\alpha\lfloor_{(X_A)_\alpha \cap (X_A)_\beta} \longrightarrow (\mathcal{G}_A)_\beta\lfloor_{(X_A)_\alpha \cap (X_A)_\beta} ,$$

which is the identity map on objects.

The notation we will use for this morphism will usually be φ^α_β but the more detailed $(\mathcal{G}_A)_{\alpha\leq\beta}$ may be used where more precision is needed. As before we write $|A|$ for X_A, the underlying set of A.

A morphism of groupoid atlases comes in several strengths as with the special case of global actions.

A *local frame* in a groupoid atlas, A, is a set $\{x_0, \cdots, x_p\}$ in a connected component of some $(\mathcal{G}_A)_\alpha$, i.e. there is some $\alpha \in \Phi_A$, $x_0, \cdots, x_p \in (X_A)_\alpha$ and arrows $g_i : x_0 \to x_i, i = 1, \cdots, p$.

A function $f : |A| \to |B|$ supports a weak morphism structure if it preserves local frames. Similar comments apply to those made above about morphisms of global actions.

The stronger form of morphism of groupoid atlases will just be called a (strong) morphism.

A *strong morphism* $\eta : A \to B$ of groupoid atlases is a pair (η_Φ, η_g) satisfying the following

$-\eta_\Phi : \Phi_A \to \Phi_B$ is a relation preserving function;

$-\eta_G : \mathcal{G}_A \to (\mathcal{G}_B)_{\eta_\Phi}$ is a (generalised) natural transformation of diagrams of groupoids **over** the function η_Φ.

To illustrate the difference between global actions and groupoid atlases, we consider some simple examples.

Example 1

Let $X = \{0, 1, 2\}, G = C_3 = \{1, a, a^2\}$ (and of course $a^3 = 1$), the cyclic group of order 3, acting by $a.0 = 1, a.1 = 2$, on X. This gives us $C_3 \ltimes X$ with 9 arrows. We set $B = C_3 \ltimes X$ as groupoid or $C_3 \curvearrowright X$ as C_3-set. We also have the example $A = C_2 \ltimes \{0, 1\} = \mathcal{I}$ considered earlier.

Both A and B will be considered initially as global actions having Φ_A and Φ_B a single element.

Any function $f : \{0, 1\} \to \{0, 1, 2\}$ supports the structure of a morphism of global actions since the only non-trivial frame in A is based on the set $x_0 = 0$, $x = 1$ and this must get mapped to a frame in B, since any non-empty subset of X is a frame in B. On the other hand, a regular morphism $\eta : A \to B$ must contain the information on a group homomorphism

$$\eta_G : C_2 \to C_3$$

which must, of course, be trivial. Hence the only regular morphism η must map all of A to a single point in B. There are thus 9 morphisms from A to B, but only 3 regular morphisms. The regular morphisms are very rigid.

Remark

It is not always the case that there are fewer regular morphisms than (general) morphisms. If A is a global action with one point and a group acting on that point and B is similar with group H, there is only one general morphism from A to B, but the set of regular morphism is 'the same as' the set of group homomorphisms from G to H.

Example 1 continued

Now consider A and B as groupoid atlases. The element $(c, 0) : 0 \to 1$ in the single groupoid determining A, must be sent to some arrow in B. The inverse of $(c, 0)$ is $(c, 1)$, so as soon as a morphism, η_G is specified on $(c, 0)$, it is determined on $(c, 1)$ since $\eta_G(c, 1) = (\eta_G(c, 0))^{-1}$. Thus if we pick an arrow in B, say,

$$(a^2, 0) : 0 \to 2,$$

we can define a morphism

$$\eta_G : A \to B$$

by specifying $\eta_G(c, 0) = (a^2, 0)$, so $\eta_G(0) = 0$, $\eta_G(1) = 2$, etc. In other words the fact that A uses an action by C_2 and B by C_3 does not inhibit the existence of morphisms from A to B. Any morphism of global actions from A to B **in this case** will support the structure of a morphism of the corresponding groupoid atlases, yet the extra structure of a "regularity condition" is supported in this latter setting. Of course the relationship between morphisms of global actions and morphisms of the corresponding groupoid atlases can be expected to be more subtle in general.

Problem/Question 1

If A and B are global actions and $f : A \to B$ is a morphism, does f support the structure of a morphism of the corresponding groupoid atlases?

In general the answer would seem to be 'no', since, if A is a global action with $\Phi_A = \{a, b : a \leq b\}$ with both X_a and X_b singlepoints, and B is similar but with Φ_A discrete, then the general morphism which corresponds to the identity does not support the structure of a (strong) morphism of the corresponding groupoid atlases because of the need for a relation function $\eta : \Phi_A \to \Phi_B$. Refining the question therefore, suppose we have a general morphism of global actions together with a relation preserving function between the coordinate systems, which is compatible with the morphism. In that case the question is related to the following question about groupoids:
if we have two groupoids A and B and a function f from the objects of A to the objects of B and which sends connected components of A to connected components of B, what obstructions are there for there to exist a functor F from A to B such that F restricted to the objects is the given f?

Clearly any global action determines a corresponding groupoid atlas as we have used above. As there are morphisms of action groupoids that do not come from regular morphisms of actions, the groupoid morphisms give a new notion of morphism of global actions. Similarly one can ask: are there "useful" groupoid atlases other than those coming from global actions? The answer is most definitely: yes.

Example 2

Let X be a set. It is well known that any equivalence relation R on X determines a groupoid with object set X. We will denote this groupoid by R as well. It is specified by

$$R(x, y) = \begin{cases} \{(x, y)\} & \text{if } xRy \\ \emptyset & \text{if } x \text{ is not related to } y. \end{cases}$$

Now suppose $R_1, \cdots, R_n$ are a family of equivalence relations on X. Then define A to have coordinate system

$$\Phi_A = \{1, \cdots, n\} \quad \text{with discrete } \leq$$

and $(G_A)_i = R_i$. This gives a groupoid atlas that will not in general be one from a global action, at least not in a natural unique way.

Example 3

Let G be a group, X a G-set and R an equivalence relation on X. Let $\Phi = \{1, 2\}$, with $\leq$ still to be specified. Take $G_1 = G \ltimes X$, $G_2 = R$ and $X_1 = X_2 = X$. Assume we have a groupoid atlas structure with this as partial data. If $\leq$ is discrete, there is no interaction between the two structures and no compatibility requirement. If $1 \leq 2$, each G-orbit is contained in an equivalence class with $\varphi_2^1(x, g) = (x, gx)$, i.e. the G-orbit structure is finer than the partition into equivalence classes. If $2 \leq 1$, the partition is finer than the orbit structure (the connected components of the groupoid G_1) and if xRy then there is some $g_{x,y} \in G$ such that $g_{x,y}x = y$.

This last case is closely related to a useful construction on global actions.

Example 4

Let $A = (X_A, G_A, \Phi_A)$ be a global action. Let $\alpha \in \Phi_A$ and $(G_A)_\alpha \curvearrowright (X_A)_\alpha$ be the corresponding action. Set R_α to be the equivalence relation determined by the $(G_A)_\alpha$-action. Thus $xR_\alpha x'$ if and only if there is some $g \in (G_A)_\alpha$ with $gx = x'$. Of course the partition of $(X_A)_\alpha$ into R_α-equivalence classes is exactly that given by the $(G_A)_\alpha$-orbits (or the $(\mathcal{G}_A)_\alpha$-components where $(\mathcal{G}_A)_\alpha$ is the corresponding groupoid).

If $\alpha \leq \beta$ then the compatibility conditions are satisfied between R_α and R_β making (X_A, R_A, Φ_A) with $R_A = \{R_\alpha : \alpha \in \Phi_A\}$ into a groupoid atlas which will be denoted $\mathrm{Equiv}(A)$.

The functions $(\mathcal{G}_A)_\alpha \to R_\alpha$ mapping the groupoid of the $(G_A)_\alpha$-action to the corresponding equivalence relation yield a natural transformation of groupoid diagrams and hence a strong morphism

$$A \to \mathrm{Equiv}(A)$$

with obvious universal properties. Of course the same construction works if A is an arbitrary groupoid atlas, that is, one not necessarily arising from a global action. The result gives a left adjoint to the inclusion of the full subcategory of atlases of equivalence relations into that of groupoid atlases. The usefulness of this construction is another reason for extending our view beyond global actions to include groupoid atlases. The notion of morphism of global actions, $f : A \to B$, translates to the notion of strong morphism, $f : Equiv(A) \to Equiv(B)$ of the corresponding groupoid atlases, at least for examples with finite orbits.

THE LINE.

We have seen that the simple action with $G = C_2$, $X = \{0, 1\}$ gives the groupoid $\mathcal{I}$ (also sometimes written [1] as it is the groupoid version of the 1-simplex). We want an analogue of a line so as to describe paths and loops. The line, L, is obtained by placing infinitely many copies of $\mathcal{I}$ end to end. It is a global action, but, as the morphisms that give paths in a global action, A, will need to be non-regular morphisms in general, it is often expedient to think of it as a groupoid atlas.

The set, $|L|$, of points of L is $\mathbb{Z}$, the set of integers; $\Phi_L = \mathbb{Z} \cup \{\square\}$, where $\square$ is an element satisfying $\square < n$ for all $n \in \mathbb{Z}$, and otherwise the relation $\leq$ is equality. (Thus $\square \leq \square$, for all $n \in \mathbb{Z}$, $\square < n$ and $n \leq n$, but that gives all related pairs.) If $n \in \Phi_L$, $(X_L)_n = \{n, n+1\}$, whilst $(X_L)_\square = |L|$ itself.

The groupoid $(\mathcal{G}_L)_n$ is a copy of $\mathcal{I}$, whilst $(\mathcal{G}_L)_\square$ is discrete with trivial vertex groups.

The underlying structure of L rests firmly on the locally finite simplicial complex structure of the ordinary real line. There the (abstract) simplicial complex structure is given by:

$$\begin{aligned} \text{Vertices } &= \mathbb{Z}, \text{ the set of integers;} \\ \text{Set of 1-simplices } &= \{\{n, n+1\} : n \in \mathbb{Z}\}, \text{ the set of adjacent pairs in } \mathbb{Z}. \end{aligned}$$

We will see shortly that there is a close link between simplicial complexes and this context of global actions/ groupoid atlases.

3 Curves, paths and connected components

Suppose A is a global action or more generally a groupoid atlas. A curve in A is simply a (weak) morphism

$$f : L \to A$$

where L is the line groupoid atlas introduced above.

This implies that $f : |L| \to |A|$ is a function so that local frames are preserved. In L the local frames are simply the adjacent pairs $\{n, n+1\}$ and the singleton sets $\{n\}$. Thus the condition that $f : L \to A$ be a path is that the sequence of points

$$\cdots, f(n), f(n+1), \cdots$$

is such that for each n, there is a $\beta \in \Phi_A$ and $g_\beta : f(n) \to f(n+1)$ in $(\mathcal{G}_A)_\beta$. (If you prefer global action notation $g_\beta \in (G_A)_\beta$ and $g_\beta f(n) = f(n+1)$.)

Note that f does not specify β and g_β, merely requiring their existence. This observation leads to a notion of a **strong curve** in A which is a morphism of groupoid atlases

$$f : L \to A$$

so for each n one gets a $\beta \in \Phi_A, \beta = \eta_\Phi(n)$ and $\eta_G : \mathcal{G}_L \to (\mathcal{G}_A)_{\eta_\Phi}$ is a natural transformation of groupoid diagrams. This condition only amounts to specifying $\eta_G(n, L) = g : f(n) \to f(n+1)$, but this time the data is part of the specification of the curve. We can thus write a strong curve as $(\cdots, f(n), g_n, f(n+1), \cdots)$, that is a sequence of points of $|A|$ together with locally defined arrows

$$g_n : f(n) \to f(n+1)$$

in the chosen local groupoid $(\mathcal{G}_A)_\beta$. Changing the β or the g_n changes the morphism. We will later see the rôle of strong curves, strong paths, etc.

A (**free**) **path** in A will be a curve that stabilises to a constant value on both its left and right ends. More precisely it is a curve $f : L \to A$ such that there are integers $N^- \leq N^+$ with the property that

$$\text{for all } n \leq N^-, \; f(n) = f(N^-);$$

$$\text{for all } n \geq N^+,\ f(n) = f(N^+).$$

We will call (N^-, N^+) a **stabilisation pair** for f.

A "based path" can be defined if A has a distinguished base point. This occurs naturally in such cases as $A = \mathbf{Gl}_n(R)$ or $A = A(G, \mathcal{H})$ for $\mathcal{H}$ a family of subgroups of a group G, but is also defined abstractly by adding the specification of the chosen base point explicitly to the data. This situation is well known from topology where a notation such as (A, a_0) would be used. We will adopt similar conventions.

If (A, a_0) is based groupoid atlas, a **based path** in (A, a_0) is a free path that stabilises to a_0 on the left, i.e., in the notation above, $f(N^-) = a_0$.

A **loop** in (A, a_0) is a based path that stabilises to a_0 on both the left and the right so $f(N^-) = f(N^+) = a_0$.

The analogue in this setting of concepts such as "connected component" should now be clear. We say that points p and q of A, a global action or groupoid atlas, are **free path equivalent** if there is a free path in A which stabilises to p on the left and to q on the right.

Clearly free path equivalence is reflexive. It is also symmetric since if $g_n : f(n) \to f(n+1)$ in a local patch then $g_n^{-1} : f(n+1) \to f(n)$. Once a free path from p to q has reached q (i.e. has stabilised at q) then it can be concatenated with a path from q to r, say, hence free path equivalence is also transitive. The equivalence classes for free path equivalence will be called **connected components**, with $\pi_0(A)$ denoting the **set of connected components** of A. If A has just one connected compent then it is said to be **connected**.

Examples

1. The prime and motivating example is the set of connected components $\pi_0\mathbf{Gl}_n(R)$ of the general linear global action.

   Suppose $x, y \in \mathbf{Gl}_n(R)$. Suppose $f : L \to \mathbf{Gl}_n(R)$ is a free path from x to y, so there are $N^- \leq N^+$ as above with

   $$\begin{aligned} &\text{if } n \leq N^-,\ f(n) = f(N^-) = x, \\ &\text{if } n \geq N^+,\ f(n) = f(N^+) = y. \end{aligned}$$

   For each $i \in [N^-, N^+]$, there is some local arrow

   $$g_i : f(n) \to f(i+1)$$

   and since $\mathbf{Gl}_n(R)$ is a global action, this means there is some $\alpha_i \in \Phi$ and $\varepsilon_i \in \mathbf{Gl}_n(R)_{\alpha_i}$ such that $\varepsilon_i f(i) = f(i+1)$. (The specification of f gives the **existence** of such an ε_i but does not actually specify which of possibly many $\varepsilon_i s$ to take, so we choose one. The choice will make no difference.) We thus have

   $$\varepsilon_{N^+}\varepsilon_{N^+-1} \cdots \varepsilon_{N^-} x = y.$$

   If $E_n(R)$ is the subgroup of elementary matrices of $Gl_n(R)$, this is the subgroup generated by all the $Gl_n(R)_\alpha$ for $\alpha \in \Phi$ and so if x and y are free path equivalent

   $$y \in E_n(R)x,$$

   i.e., x and y are in the same right coset of $E_n(R)$.

   Conversely if $y \in E_n(R)x$, there is an element $\varepsilon \in E_N(R)$ such that $y = \varepsilon x$, but ε can be written (in possibly many ways) as a product of elementary matrices

   $$\varepsilon = \varepsilon_N \cdots \varepsilon_1$$

   with $\varepsilon_i = Gl_n(R)_{\alpha_i}$, say. Then defining

   $$f : L \to \mathbf{Gl}_n(R)$$

   by

   $$f(n) = \begin{cases} x & n \leq 0 \\ \varepsilon_n \cdots \varepsilon_1 x & 1 \leq n \leq N \\ y & n \geq N \end{cases}$$

gives a free path from x to y in $\mathbf{Gl}_n(R)$.

Thus $\pi_0(\mathbf{Gl}_n(R)) = Gl_n(R)/E_n(R)$, the set of right cosets of $Gl_n(R)$ modulo elementary matrices. This is, of course, the algebraic K-group $K_1(n, R)$ if R is a commutative ring.

We can naturally ask the question: 'is $K_2(n, R) \cong \pi_1(\mathbf{Gl}_n(R))$?' even if we have not yet defined the righthand side of this.

Note the use of the strong rather than the weak version of paths would not change the resulting π_0.

2. Suppose $A = A(G, \mathcal{H})$. Can one calculate $\pi_0(A)$? A similar argument to that in 1 above shows that if $x, y \in |A| = |G|$, then they are free path equivalent if and only if there are indices $\alpha_i \in \Phi$ and elements $h_{\alpha_i} \in H_{\alpha_i}$, such that

$$h_{\alpha_n} \cdots h_{\alpha_0} x = y$$

for some n. Thus writing $\langle \mathcal{H} \rangle = \langle H_i : i \in \Phi \rangle$ for the subgroup of G generated by the family $\mathcal{H} = \{H_i : i \in \Phi\}$, we clearly have

$$\pi_0(A(G, \mathcal{H})) = G/\langle \mathcal{H} \rangle.$$

Again the question arises as to $\pi_1(A(G, \mathcal{H}))$: what is it and what does it tell us?

4 Conclusion

In such a short paper we can not do more than scratch the surface of the theory. A much more extensive set of notes is in preparation with a preliminary version available, (see preprint 99.27 at www.informatics.bangor.ac.uk/public/mathematics/research/preprints/99/algtop99.html).

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