On two non-discrete localic generalizations of π_0

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In celebration of the 100th anniversary of the birth of Charles Ehresmann, a visionary mathematician

Introduction

Charles Ehresmann [8] introduced local structures in the 1950's mainly for two reasons, as he explains in [9]. The first was that, as he observed, it is not the points of a space which are important in arguments involving localization, but its open parts. The second reason was that, for infinitesimal structures of various kinds, there is always an underlying local structure.

Central in the algebraic topology for toposes is the notion of a set of connected components of a topos. In topos theory over a base topos $\mathscr{S}$, any locally connected topos $\mathscr{F}$ bounded over $\mathscr{S}$ with structure morphism $f : \mathscr{F} \longrightarrow \mathscr{S}$ has a (discrete, localic) topos $\Pi_0(\mathscr{F})$ of connected components, to wit, the topos $\mathscr{S}/f_!(1)$. The purpose of this paper is to discuss (non-discrete) localic generalizations of $\Pi_0(\mathscr{F})$ in the absence of local connectedness.

The first generalization of $\Pi_0(\mathscr{F})$ (section 1) is the topos $\mathscr{P}_0(\mathscr{F})$ of path components of $\mathscr{F}$. It was presented at the Workshop in the Ramifications of Category Theory, Firenze, 2003 [3]. The construction of $\mathscr{P}_0(\mathscr{F})$ is obtained as an instance of a general pushout construction of collapsing paths to a point, first used in Synthetic Differential Geometry [17] in the case of infinitesimal paths. The totally disconnected topos $\mathscr{P}_0(\mathscr{F})$ leads in turn to a natural definition of the paths fundamental groupoid $\Pi_1^{(\mathrm{path})}(\mathscr{F})$ of $\mathscr{F}$, and to a simple comparison map from it to its coverings fundamental groupoid $\Pi_1^{(\mathrm{cov})}(\mathscr{F})$. It is of particular interest when $\Pi_1(\mathscr{F})^{(\mathrm{path})}$ is a totally disconnected localic groupoid.

The second generalization to $\Pi_0(\mathscr{F})$ (section 2) is the zero-dimensional topos $\Pi_\bullet(\mathscr{F})$ of quasicomponents of $\mathscr{F}$. It presents results obtained in joint work with Jonathon Funk and included, with proofs and additional material, in [5]. We view the construction of $\Pi_\bullet(\mathscr{F})$ as an instance of a new factorization of a geometric morphism into a hyperpure geometric morphism followed by a complete spread. This new factorization generalizes a result of [4], established for the case when the domain topos is locally connected. In order to achieve it without local connectedness, we follow the lead of Fox [10] and Michael [14] in topology. The topos $\Pi_\bullet(\mathscr{F})$ is localic and zero-dimensional.

1 The topos of path components

The results of this section were presented at the Workshop in the Ramifications of Category Theory, Firenze, 2003 [3].

A topos $\mathscr{F}$ over $\mathscr{S}$ is said to be *totally disconnected* if the "constant paths" geometric morphism $c_{\mathscr{F}} : \mathscr{F} \longrightarrow \mathscr{F}^I$ is a surjection, where I denotes the topos of sheaves on the unit interval regarded as a locale. As shown in [12], $c_{\mathscr{F}}$ is an inclusion, so that in effect, for a totally paths disconnected topos, $c_{\mathscr{F}}$ is an equivalence.

Recall also that a topos $\mathscr{F}$ is said to be *path-connected* if $f : \mathscr{F} \longrightarrow \mathscr{S}$ and the evaluation-at-the-endpoints map $\varepsilon : \mathscr{F}^I \longrightarrow \mathscr{F} \times_{\mathscr{S}} \mathscr{F}$ are both open surjections. The following is shown in [12], using also the main result from [16].

Lemma 1.1 *Any path-connected geometric morphism is orthogonal to any totally disconnected geometric morphism. Any connected locally connected geometric morphism is path-connected.*

The bipushout $(*)$ below is an instance of a general construction having as a notable example the topos of A-discrete objects in a given topos $\mathscr{F}$, where A is an $A.T.O.$ [13]. It was investigated in [17]. We apply it in the 2-category $\mathbf{Top}_{\mathscr{S}}$ of $\mathscr{S}$-bounded toposes, geometric morphisms and 2-cells of such, with the object I of sheaves on the unit interval locale, which is exponentiable as well as connected (an $A.T.O.$ is furthermore projective).

We take the bipushout in $\mathbf{Top}_{\mathscr{S}}$:

$$\begin{array}{ccc} \mathscr{F}^I \times I & \xrightarrow{p_0} & \mathscr{F}^I \\ \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \zeta_{\mathscr{F}}} \\ \mathscr{F} & \xrightarrow[\eta_{\mathscr{F}}]{} & \mathscr{P}_0(\mathscr{F}) \end{array} \qquad (*)$$

and call $\mathscr{P}_0(\mathscr{F})$ the *topos of paths components* of $\mathscr{F}$.

The construction of the topos of paths components of a topos $\mathscr{F}$ leads to a definition of its paths fundamental groupoid without any assumptions.

Proposition 1.2 *The diagram* $(**)$ *given by*

$$\mathscr{P}_0(\mathscr{F}^I) \times_{\mathscr{P}_0(\mathscr{F})} \mathscr{P}_0(\mathscr{F}^I) \; \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \; \mathscr{P}_0(\mathscr{F}^I) \; \begin{array}{c} \xrightarrow{\mathscr{P}_0(\varepsilon_0)} \\ \longleftarrow \\ \xrightarrow[\mathscr{P}_0(\varepsilon_1)]{} \end{array} \; \mathscr{P}_0(\mathscr{F})$$

is a groupoid in $\mathbf{Top}_{\mathscr{S}}$*, with operations induced by the usual operations of composition and inversion of paths.*

Proof. An argument to show that it is a groupoid as indicated can be made analogously to that of [12] (Proposition 4.3). □

Definition 1.3 *The colimit of the groupoid topos given in Proposition 1.2 is said to be the paths fundamental groupoid of* $\mathscr{F}$ *and denoted* $\Pi_1{}^{(path)}(\mathscr{F})$.

The pushout $(*)$ is also the basis for the *paths totally disconnected reflection* under suitable assumptions.

Definition 1.4 *A topos* $\mathscr{F}$ *bounded over* $\mathscr{S}$ *is said to be* paths-tight *if the evaluation morphism* $\mathscr{F}^I \times I \xrightarrow{\text{ev}} \mathscr{F}$ *is locally connected. We say that* $\mathscr{F}$ *is* homotopies-tight *if* $\mathscr{F}^I$ *is paths-tight.*

Theorem 1.5 *Let* $\mathscr{F}$ *be paths-tight. Then in the bipushout* $(*)$, *the following hold:*

1. *The unit* $\eta_{\mathscr{F}}$ *is connected and locally connected.*

2. *For any totally disconnected topos* $\mathscr{X}$, $\eta_{\mathscr{F}}$ *induces a bijection*

$$\mathbf{Top}_{\mathscr{S}}(\mathscr{P}_0(\mathscr{F}), \mathscr{X}) \cong \mathbf{Top}_{\mathscr{S}}(\mathscr{F}, \mathscr{X}).$$

3. $\mathscr{P}_0(\mathscr{F})$ *is a totally disconnected topos.*

4. $\mathscr{P}_0(\mathscr{F})$ *is paths-tight.*

5. *If* $\mathscr{F}$ *is also homotopies-tight, then the groupoid* $(**)$ *of Proposition 1.2 is totally disconnected.*

Proof.

1. That $\eta_{\mathscr{F}}$ is locally connected follows from the fact that in the bipushout defining $\mathscr{P}_0(\mathscr{F})$, both ev and p_0 are locally connected. Furthermore, p_0 is connected (since the locale I is connected) hence so is $\eta_{\mathscr{F}}$.

2. Given $\mathscr{F} \xrightarrow{\varphi} \mathscr{X}$, there is a commutative diagram

$$\begin{array}{ccc} \mathscr{F}^I \times I & \xrightarrow{p_0} & \mathscr{F}^I \\ \downarrow{\scriptstyle \varphi\cdot\text{ev}} & & \downarrow{\scriptstyle \gamma_{\mathscr{F}^I}} \\ \mathscr{X} & \xrightarrow[\gamma_{\mathscr{X}}]{} & \mathscr{S} \end{array} \tag{1}$$

   with top horizontal connected and locally connected, and bottom horizontal paths totally disconnected hence, by Lemma 1.1,there is a unique diagonal fill-in $\mathscr{F}^I \xrightarrow{\psi} \mathscr{X}$. From the bipushout property now follows the existence of a unique $\mathscr{P}_0(\mathscr{F}) \xrightarrow{\kappa_{\mathscr{F}}} \mathscr{X}$ such that $\varphi \cong \kappa_{\mathscr{F}} \cdot \eta_{\mathscr{F}}$.

3. One shows first that $c_{\mathscr{P}_0(\mathscr{F})}\zeta_{\mathscr{F}} \cong \eta_{\mathscr{F}}{}^I$. By a result of [15] $\eta_{\mathscr{F}}{}^I$ is a stable (in fact, an open) surjection. It follows that $c_{\mathscr{P}_0(\mathscr{F})}$ is a surjection.

4. We wish to show that the evaluation morphism

$$\mathscr{P}_0(\mathscr{F})^I \times I \xrightarrow{\text{ev}} \mathscr{P}_0(\mathscr{F})$$

is locally connected. This follows from the commutativity of the square

$$\begin{array}{ccc} \mathscr{F}^I \times I & \xrightarrow{\eta_{\mathscr{F}}{}^I \times I} & \mathscr{P}_0(\mathscr{F})^I \times I \\ \Big\downarrow \text{ev} & & \Big\downarrow \text{ev} \\ \mathscr{F} & \xrightarrow[\gamma_{\mathscr{F}}]{} & \mathscr{P}_0(\mathscr{F}) \end{array} \qquad (2)$$

and the facts that the composite of the left vertical with the bottom horizontal is locally connected, while the top horizontal is an open surjection as it is the bipullback of an I-exponential of a connected locally connected morphism.

5. Basic properties of totally disconnected geometric morphisms imply that the structure maps in the diagram $(**)$ are totally disconnected.

□

Definition 1.6 *We say that a topos* $\mathscr{F}$ has a locale of paths components *if the topos* $\mathscr{P}_0(\mathscr{F}) \longrightarrow \mathscr{S}$ *is localic.*

Corollary 1.7 *Let* $\mathscr{F}$ *and* $\mathscr{F}^I$ *be both paths-tight and have a locale of paths components. Then,* $\Pi_1{}^{(path)}(\mathscr{F})$ *is the classifying topos of a totally disconnected localic groupoid.*

Let us revisit briefly the coverings fundamental groupoid of a topos in the locally connected case. First note that if $\mathscr{F}$ is locally connected and paths-tight over $\mathscr{S}$, then necessarily $\mathscr{P}_0(\mathscr{F}) \cong \mathscr{S}/f_!(1)$, since any locally connected geometric morphism which is at the same time totally disconnected must be a local homeomorphism. In that case, the totally disconnected groupoid $(**)$ is discrete. We record the following fact from [2].

Proposition 1.8 *If* $\mathscr{F}$ *is locally connected and locally simply connected with Galois object* A*, there is an isomorphism*

$$\text{Aut}(A) \cong f_!(A \times A)$$

of discrete groupoids, with the right-hand side the Galois groupoid in the sense of Janelidze [11].

Lemma 1.9 *Let* $\mathscr{F}$ *be locally connected and locally simply connected with Galois object* A*. Then, the induced geometric morphism*

$$\mathscr{P}_0(\mathscr{F}/A) \longrightarrow \mathscr{P}_0(\mathscr{F})$$

is an equivalence.

Proof. If A is a Galois object then $\mathscr{F}/A \longrightarrow \mathscr{F}$ is connected locally connected, hence orthogonal to totally disconnected. □

In view of Lemma 1.9, we shall regard the Galois topos of a locally connected and locally simply connected topos $\mathscr{F}$ with Galois object A as the colimit $\Pi_1{}^{(cov)}(\mathscr{F})$ of the totally disconnected (in fact, discrete) localic groupoid in $\mathbf{Top}_{\mathscr{S}}$, with object of objects $\mathscr{P}_0(\mathscr{F}/A \times_{\mathscr{S}} \mathscr{F}/A)$.

Definition 1.10 *We say that* $\mathscr{F}$ *has the* unique paths-lifting property *if the geometric morphism* $\mathscr{F}^I \xrightarrow{\varepsilon} \mathscr{F} \times_{\mathscr{S}} \mathscr{F}$ *is connected locally connected.*

Proposition 1.11 *Let* $\mathscr{F}$ *be a locally connected and locally simply connected topos over* $\mathscr{S}$ *with Galois object A. Then, the following hold.*

1. *There is a canonical groupoid homomorphism*

$$\Pi_1{}^{(path)}(\mathscr{F}) \xrightarrow{\sigma_{\mathscr{F}}} \Pi_1{}^{(cov)}(\mathscr{F})$$

in $\mathbf{Top}_{\mathscr{S}}$.

2. *If in addition* $\mathscr{F}$ *is assumed paths-tight, homotopies-tight, and to have the unique paths-lifting property, then the groupoid topos homomorphism* $\sigma_{\mathscr{F}}$ *is an equivalence.*

Proof. The homomorphism $\sigma_{\mathscr{F}}$ is given by the following diagram in $\mathbf{Top}_{\mathscr{S}}$.

$$\begin{array}{ccc}
\mathscr{P}_0((\mathscr{F}/A)^I) \times_{\mathscr{P}_0(\mathscr{F}/A)} \mathscr{P}_0((\mathscr{F}/A)^I) & \longrightarrow & \mathscr{P}_0(\mathscr{F}/A \times_{\mathscr{S}} \mathscr{F}/A) \times_{\mathscr{P}_0(\mathscr{F}/A)} \mathscr{P}_0(\mathscr{F}/A \times_{\mathscr{S}} \mathscr{F}/A)) \; . \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
\mathscr{P}_0((\mathscr{F}/A)^I) & \longrightarrow & \mathscr{P}_0(\mathscr{F}/A \times_{\mathscr{S}} \mathscr{F}/A) \\
\downarrow\downarrow & & \downarrow\downarrow \\
\mathscr{P}_0(\mathscr{F}/A) & \underset{\mathrm{id}}{\longrightarrow} & \mathscr{P}_0(\mathscr{F}/A)
\end{array}$$

It follows easily that if $\mathscr{F}$ has the unique paths-lifting property in the sense of Definition 1.10, the homomorphism of groupoids in $\mathbf{Top}_{\mathscr{S}}$ is an equivalence. □

2 The topos of quasicomponents

The results from this section are joint with J. Funk and are all contained in [5].

For a geometric morphism $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ with a locally connected domain $f : \mathscr{F} \longrightarrow \mathscr{S}$, there is [4] a unique factorization $\mathscr{F} \longrightarrow \mathscr{Y} \longrightarrow \mathscr{E}$ into a pure (dense) geometric morphism followed by a complete spread (with a locally connected domain). The middle geometric morphism $\mathscr{Y} \xrightarrow{\varphi} \mathscr{E}$ associated with (ψ, f) is the topos of cogerms of components of the image of ψ^* in $\mathscr{F}$. This is the topos-theoretic version of the complete spreads in the sense of Fox [10]. If $\mathscr{E}$ is equal to $\mathscr{S}$ and $\psi = f$, then this topos is just $\Pi_0(\mathscr{F}) = \mathscr{S}/f_!(1)$ and it is therefore discrete.

We now seek to generalize this to the non-locally connected case. In topology, E. Michael [14] has shown how to do this by considering cogerms of quasicomponents instead of cogerms of components, and then by identifying the first factor in order to establish uniqueness. We begin by recalling the notion of a definable morphism from [1]. It is the constructive version of the notion of (sum of) clopen in topology. The notion of a definable dominance is from [6].

Definition 2.1 *A morphism* $X \xrightarrow{m} Y$ *in a topos* $\mathscr{F}$ *is* definable *if it can be put in a pullback square as follows.*

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ \downarrow & & \downarrow \\ f^*A & \xrightarrow[f^*n]{} & f^*B \end{array}$$

A definable subobject *is a monomorphism that is definable.*

Definable morphisms do not compose in general, not even over a Boolean topos. Denote the characteristic map of $f^*(\top) : f^*1 \longrightarrow f^*(\Omega_{\mathscr{S}})$ by $\tau : f^*\Omega_{\mathscr{S}} \longrightarrow \Omega_{\mathscr{F}}$.

Definition 2.2 *f is said to be* a definable dominance *if τ is a monomorphism (subopen), and if definable morphisms compose.*

We shall assume in the rest of this section that $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ is an arbitrary geometric morphism over $\mathscr{S}$, with $f : \mathscr{F} \longrightarrow \mathscr{S}$ a definable dominance. Any locally connected topos is a definable dominance.

The notion of a ψ-cover was introduced in [7]. We need the notion of a weak ψ-cover in order to describe our main construction here.

Definition 2.3 A *ψ-cover* is a diagram

$$\begin{array}{ccc} V & \twoheadrightarrow & U \\ \downarrow & & \downarrow \\ \psi^*E & \longrightarrow & \psi^*C \\ \downarrow & & \downarrow \\ f^*A & \longrightarrow & f^*B \end{array}$$

in $\mathscr{F}$, where $V \rightarrowtail \psi^*E$ and $U \rightarrowtail \psi^*C$ are definable subobjects. If the square

$$\begin{array}{ccc} E & \longrightarrow & C \\ \downarrow & & \downarrow \\ e^*A & \longrightarrow & e^*B \end{array}$$

coming from $\mathscr{E}$ is a pullback, then we call the diagram a *weak ψ-cover.*

Suppose that $\mathscr{E} = Sh(J, \mathbf{C})$. Let $\mathbf{H}$ denote the category of pairs (C, U), such that $U \rightarrowtail \psi^*(h_C)$ is a definable subobject in $\mathscr{F}$, where $h : \mathbf{C} \longrightarrow \mathscr{E}$ denotes the Yoneda functor (plus sheafification). Let H denote $\psi_*(f^*\Omega_{\mathscr{S}})$ in $\mathscr{E}$. A morphism $h_C \xrightarrow{U} H$ in $\mathscr{E}$ corresponds to such an object (C, U). We have a geometric morphism $\mathscr{F} \xrightarrow{\zeta} P(\mathbf{H})$ such that $\zeta^*(h_{(C,U)}) = U$, where again $h : \mathbf{H} \longrightarrow P(\mathbf{H})$ denotes another Yoneda functor.

If $\mathscr{F}$ is locally connected, there is a category of components $\mathbf{Y}$, whose objects are pairs (C, α) such that $\alpha \rightarrowtail \psi^*(h_C)$ is a connected component, and which is a full subcategory of $\mathbf{H}$. In the absence of local connectedness there is no category of components $\mathbf{Y}$. Nevertheless, there is a subtopos $\mathscr{Z} \rightarrowtail P(\mathbf{H})$ (generalizing $P(\mathbf{Y})$) of sheaves for the topology in $\mathbf{H}$ generated by the weak ψ-covers,

Form the bipullback to $\mathscr{E}$ producing the following diagram of geometric morphisms.

$$
\begin{array}{ccccc}
\mathscr{F} & \xrightarrow{\rho} & \mathscr{X} & \rightarrowtail & \mathscr{Z} \\
 & \searrow^{\psi} & \downarrow & \lrcorner & \downarrow \tau \\
 & & \mathscr{E}^{H^{\mathrm{op}}} & \rightarrowtail & P(\mathbf{H}) \\
 & & \downarrow & \lrcorner & \downarrow \\
 & & \mathscr{E} & \rightarrowtail & P(\mathbf{C})
\end{array} \tag{3}
$$

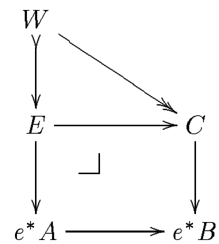
Thus, we have a factorization $\psi \cong \varphi \cdot \rho$, where φ denotes the composite $\mathscr{X} \rightarrowtail \mathscr{E}^{H^{\mathrm{op}}} \longrightarrow \mathscr{E}$. We shall refer to the middle topos $\mathscr{X}$ as *the Michael topos of ψ* and to the factorization as *the Michael factorization of ψ.*

Next, we identify the first factors in the Michael factorization of ψ. We do this by translating into topos theory a condition (B) from [14], itself replacing a condition (A) from [10].

Definition 2.4 *$\mathscr{F} \xrightarrow{\rho} \mathscr{E}$ is said to be* hyperpure *if every diagram*

$$
\begin{array}{ccc}
V & & \\
\downarrow & \searrow & \\
\rho^* E & \longrightarrow & \rho^* C \\
\downarrow & \lrcorner & \downarrow \\
f^* A & \longrightarrow & f^* B
\end{array}
$$

in which $V \rightarrowtail \rho^*E$ is definable is given locally by a diagram



from $\mathscr{E}$, where $W \rightarrowtail E$ is definable, subject to a uniqueness requirement.

Remark 2.5 We have the implications

$$\text{connected} \Rightarrow \text{hyperpure} \Rightarrow \text{direct image preserves } \mathscr{S}\text{-coproducts} \\ \Rightarrow \text{pure (dense)} \Rightarrow \text{dense.}$$

The key result in this section is the following. We state it without proof, but see [5] as for everything else in this section.

Proposition 2.6 *The first factor of the Michael factorization is hyperpure.*

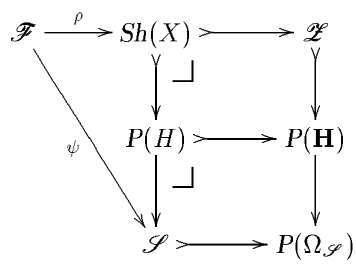
It remains to identify the second factors in the Michael factorization. They are best described by means of a *cover refinement property* in a manner analogous to that of [7]. This description is not as transparent as in the locally connected case, but in topology it amounts to the property that cogerms of quasicomponents converge. They will be called *complete spreads.*

Theorem 2.7 *The Michael factorization of a geometric morphism $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ (whose domain $f : \mathscr{F} \longrightarrow \mathscr{S}$ is a definable dominance) into a hyperpure geometric morphism ρ followed by a complete spread φ (with domain a definable dominance) is unique up to equivalence.*

Proof. The assumption that ρ is hyperpure, hence pure (dense) implies that it induces an equivalence between $\mathbf{H}_{<f,\psi>}$ and $\mathbf{H}_{<x,\varphi>}$, which we may simply denote by $\mathbf{H}$. Next, we need to know that there is also an equivalence between the topology on $\mathbf{H}$ determined by the weak ψ-covers, and that determined by the weak φ-covers. This needs the full import of Definition 2.4 and it is straightforward. □

We end with an analysis of the special case where the codomain topos $\mathscr{E}$ and the base topos $\mathscr{S}$ agree, with $\psi = f$. This particular case gives the locale of quasicomponents of a topos.

The object classifier $\Omega_{\mathscr{S}}$ serves as a site for $\mathscr{S}$, and $H = f_* f^* \Omega_{\mathscr{S}}$. We have the following topos pullbacks.



For example, if $\mathscr{S} = Set$, then $\mathbf{H}$ is the lattice of all complemented subobjects $U \rightarrowtail 1_{\mathscr{F}}$ plus an extra object $(0,0)$ that is below the complemented subobject $0 \rightarrowtail 1_{\mathscr{F}}$. H is the same lattice as $\mathbf{H}$, except without the extra object $(0,0)$. The coverage in H giving the locale X is simply the $\mathscr{F}$-join coverage: $\{U_a \rightarrowtail U\}$ covers just when $\bigvee U_a = U$ in the lattice of subobjects of $1_{\mathscr{F}}$ in $\mathscr{F}$.

The frame $\mathscr{O}(X)$ can be identified as the lattice of subobjects $V \rightarrowtail 1_{\mathscr{F}}$ that are joins of complemented ones $U \rightarrowtail 1_{\mathscr{F}}$. It follows that X is a zero-dimensional locale in the sense that $Sh(X) \longrightarrow \mathscr{S}$ is a spread [4]. Moreover, this construction is a reflection (left adjoint) of locales into zero-dimensional locales. Completeness is not an issue here – the structure map of a topos is automatically complete.

A point $1 \longrightarrow X$ is a filter (upclosed and closed under finite meets) of complemented subobjects of $1_{\mathscr{F}}$ that is inaccessible by the $\mathscr{F}$-joins. We interpret such a filter as *a quasicomponent of* $\mathscr{F}$. In topology, this agrees with the usual notion of quasicomponent.

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