A FEW POINTS ON DIRECTED ALGEBRAIC TOPOLOGY

by Marco GRANDIS (*)

Dedicated to Charles Ehresmann, on the centennial of his birth

Abstract. Directed Algebraic Topology is a recent field, deeply linked with Category Theory. A 'directed space' has directed homotopies (generally non reversible), directed homology groups (enriched with a preorder) and fundamental *n-categories* (replacing the fundamental n-groupoids of the classical case). Applications have been mostly developed in the theory of concurrency. Unexpected links with noncommutative geometry and the modelling of biological systems have emerged.

1. Introduction. Directed Algebraic Topology (DAT) studies 'directed spaces' in some sense, where paths and homotopies cannot generally be reversed; for instance: simplicial and cubical sets, ordered topological spaces, 'spaces with distinguished paths', 'inequilogical spaces', etc. Its present applications deal mostly with the analysis of concurrent processes (see [Go, FRGH] and references there), but its natural range should cover non reversible phenomena, in any domain.

Here, after a review of a series of papers devoted to this subject ([G4] to [G8]), we shall give some hints at future developments and new interactions with other domains. A wider literature can be found in the papers mentioned above.

Directed spaces can be studied with directed versions of the classical tools of Algebraic Topology. Thus, the *directed homology groups* $\uparrow H_n(X)$ (Sections 2-3, [G5, G6]) are *preordered* abelian groups. Similarly, the *fundamental category* $\uparrow \Pi_1(X)$ (Sections 4-5, [G4, G7]) replaces the classical fundamental groupoid; it also allows one to study situations where all directed loops are trivial (so that the fundamental monoids are trivial, and $\uparrow H_1(X)$ is reduced to its algebraic part). The study of higher fundamental categories has begun, in a strict 2-dimensional version [G8]; but here we prefer to anticipate a work in preparation on a *lax* version, which seems to be more natural and adapted to higher extensions (Section 6).

DAT has thus a deep interaction with ordinary and higher dimensional category theory, clearer than classical Algebraic Topology. Natural links with Differential Geometry are being studied, while unexpected connections with Noncommutative Geometry have already been developed (Section 3, [G5, G6]); these deal with parallel realisations - in Noncommutative Geometry and DAT - of orbit spaces and spaces of leaves which are trivial in ordinary topology. Other interactions have recently

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appeared, between the notion of *root* of a category developed by A.C. Ehresmann [Eh] for modelling biological systems, and our study of the fundamental category (Section 7).

We shall end with a review of topological settings for DAT (Section 8) and a first step in providing a formal setting (Section 9), based on Kan ideas (abstracting the (co)cylinder functor, cf. [Ka]) rather than on Quillen model structures, which do not seem to be able to formalise privileged directions and directed homotopies.

This paper contains my contribution at the conference "Charles Ehresmann: 100 ans", Amiens, 7-9 October 2005. I am particularly grateful for this opportunity, since the position of Directed Algebraic Topology at the confluence of Topology, Geometry and Category Theory, can presumably be viewed as coherent with the research lines pursued by Charles and Andrée C. Ehresmann.

2. Combinatorial settings and directed homology. First, directed homotopy and homology can be developed for cubical sets (as in [G5]) and simplicial sets.

Let us recall that a topological space T has *intrinsic symmetries*, appearing - at the lowest level - in the reversion of its paths. Thus, the set $\Delta_n T = \mathbf{Top}(\Delta^n, T)$ of its singular simplices inherits from the standard simplex Δ^n an obvious action of the symmetric group S_{n+1}, while the set $\square_n T = \mathbf{Top}([0, 1]^n, T)$ of its singular cubes has a similar action of the hyperoctahedral group (the group of symmetries of the n-cube). These combinatorial structures produce the singular homology of the space T, which can be equivalently defined as the homology of the chain complex associated to the simplicial set ΔT, or the homology of the (normalised) chain complex associated to the cubical set $\square T$. Here, *a specific motivation for preferring cubical sets will be our use of the natural order on* $\mathbf{I}^n$ (cf. Section 4).

Now, *bypassing topological spaces*, an abstract cubical set X is a merely combinatorial structure, consisting of a sequence of sets X_n, with faces ∂_i^α: $X_n \to X_{n-1}$ and degeneracies e_i: $X_{n-1} \to X_n$ ($\alpha = \pm$; $i = 1,..., n$) satisfying the well-known cubical relations. This structure has been used in two ways, in [G5]: to break the symmetries considered above and to perform constructions, namely quotients, which would be useless in ordinary topology.

For the first aspect, note that an 'edge' in X_1 need not have any counterpart with reversed vertices, nor a 'square' in X_2 any counterpart with horizontal and vertical faces interchanged. Thus, our structure has 'privileged directions' in any dimension (classically ignored), and the (usual) combinatorial homology of X can be given a preorder, generated by taking the given cubes as positive. For instance, the obvious cubical model $\uparrow\mathbf{s}^n$ of the n-dimensional sphere, with one non-degenerate cube in dimension n, has *directed homology* $\uparrow H_n(\uparrow\mathbf{s}^n)$ consisting of the group of integers, *with the natural order*. Direction should not be confused with orientation, as shown by the model $\uparrow\mathbf{t}^2 = \uparrow\mathbf{s}^1 \otimes \uparrow\mathbf{s}^1$ of the torus, where $\uparrow H_1(\uparrow\mathbf{t}^2) \cong \uparrow\mathbf{Z}^2$ has the *product order*.

Secondly, it may happen that a quotient $T/\sim$ of a topological space has a trivial topology, while the corresponding quotient of its singular cubical set $\square T$ keeps a relevant topological information, detected by its homology and agreeing with the interpretation of such quotients in noncommutative geometry, as recalled below.

3. Interactions with noncommutative geometry. Let us start from the well-known *irrational rotation* C*-algebras, also known as 'noncommutative tori'.

First, take the line $\mathbf{R}$ and its (dense) additive subgroup $G_\vartheta = \mathbf{Z} + \vartheta\mathbf{Z}$ (ϑ irrational) acting on the former by translations. In **Top**, the orbit space $\mathbf{R}/G_\vartheta = \mathbf{S}^1/\vartheta\mathbf{Z}$ is trivial: an uncountable set with the coarse topology. Second, consider the *Kronecker foliation* F of the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, with irrational slope ϑ, and the set $\mathbf{T}^2_\vartheta = \mathbf{T}^2/\equiv_F$ of its leaves (in bijective correspondence with the previous set $\mathbf{R}/G_\vartheta$). Again, topology gives no information on $\mathbf{T}^2_\vartheta$, since all leaves are dense and the quotient space $\mathbf{T}^2/\equiv_F$ is coarse.

In noncommutative geometry, both these sets are 'interpreted' as the (noncommutative) C^*-algebra A_ϑ, generated by two unitary elements u, v under the relation $vu = \exp(2\pi i\vartheta).uv$, and called the *irrational rotation algebra* associated with ϑ, or also a *noncommutative torus* [Ri, Co]. Both its complex K-theory groups are two-dimensional. These algebras have been classified, by proving that $K_0(A_\vartheta) \cong \uparrow G_\vartheta$ *as a (totally) ordered subgroup of* $\mathbf{R}$. Thus, A_ϑ and $A_{\vartheta'}$ are *strongly Morita equivalent* if and only if $\uparrow G_\vartheta \cong \uparrow G_{\vartheta'}$, if and only if ϑ and ϑ' are equivalent modulo the action of the group $PGL(2, \mathbf{Z})$ [PV, Ri].

For a group G acting *properly* on an acyclic space T, a classical result says that the homology of the orbit space T/G is isomorphic to the homology of the group G; these results can be extended to *free* actions if we replace T with its singular cubical set $\square T$ and take the *quotient cubical set* $(\square T)/G$ ([G5], Thm. 3.3). Thus, the trivial orbit space $\mathbf{R}/G_\vartheta$ can be replaced with a non-trivial cubical set, $(\square \mathbf{R})/G_\vartheta$, whose homology is the same as the homology of the group $G_\vartheta \cong \mathbf{Z}^2$, and coincides with the homology of the torus $\mathbf{T}^2$. *Algebraically*, all this is in accord with the noncommutative C^*-algebra A_ϑ, but our result is independent of ϑ and does not allow us to recover it.

Now, this similarity can be enhanced. The quotient $(\square \mathbf{R})/G_\vartheta$ can be modified, replacing $\square \mathbf{R}$ with the cubical set $\square \uparrow\mathbf{R}$ of the *directed* line, formed of all *order-preserving* maps $\mathbf{I}^n \to \mathbf{R}$. Algebraically, the homology groups are unchanged, but now $\uparrow H_1(\uparrow\mathbf{R}/G_\vartheta) \cong \uparrow G_\vartheta$ as an ordered subgroup of $\mathbf{R}$ ([G5], Thm. 4.8): thus the *rotation cubical sets* $C_\vartheta = \uparrow\mathbf{R}/G_\vartheta$ have the same classification *up to isomorphism* ([G5], Thm. 4.9) as the rotation C^*-algebras A_ϑ *up to strong Morita equivalence*, and ϑ is determined up to the action of $PGL(2, \mathbf{Z})$. This example shows that *the ordering of directed homology can carry a relevant information*. Further, comparison with the stricter classification of the algebras A_ϑ *up to isomorphism* shows that cubical sets provide a sort of 'noncommutative topology', without the metric character of noncommutative geometry.

(The *inequilogical spaces* $C'_\vartheta = (\uparrow\mathbf{R}, \equiv_{G_\vartheta})$ give the same results, cf. Section 8.)

4. A basic topological setting. After the combinatorial setting considered above, the simplest *topological* situation where one can study directed paths and directed homotopies is likely the category p**Top** of *preordered topological spaces* and *preorder-preserving continuous mappings*. (A *preorder* relation is reflexive and transitive; it is called an *order* if it is also anti-symmetric.)

In this setting, a (directed) *path* of the preordered space X is a morphism $a\colon \uparrow[0,1] \to X$, defined on the standard directed interval $\uparrow\mathbf{I} = \uparrow[0, 1]$ (with euclidean topology and natural order). A (directed) *homotopy* $\varphi\colon f \to g\colon X \to Y$, *from* f *to* g, is a map $\varphi\colon X\times\uparrow\mathbf{I} \to Y$ coinciding with f on the lower basis of the *cylinder* $X\times\uparrow\mathbf{I}$, with g on the upper one. Of course, this (directed) cylinder is a product in p**Top**: it is equipped with the product topology *and* with the product preorder, where $(x, t) \prec (x', t')$ if $x \prec x'$ in X and $t \leq t'$ in $\uparrow\mathbf{I}$.

The category p**Top** has all limits and colimits, constructed as in **Top** and equipped with the initial or final preorder for the structural maps. The forgetful functor $U\colon$ p**Top** $\to$ **Top** with values in the category of topological spaces has both a left and a right adjoint, $D \dashv U \dashv C$, where DX (resp. CX) is the space X with the *discrete* order (resp. the *coarse pre*order). The standard embedding of **Top** in p**Top** will be the *coarse* one, so that all (ordinary) paths in X are directed in CX. *Note that the category of* ordered *spaces does not allow for such an embedding*, *Note that the category of* ordered *spaces does not allow for such an embedding*, and would not allow one to view classical Algebraic Topology within the Directed one.

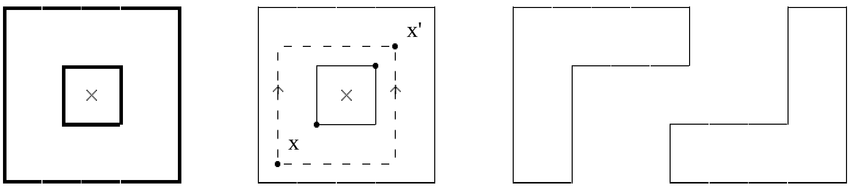
The fundamental category $C = \uparrow\Pi_1(X)$ has, for arrows, the classes of directed paths up to the equivalence relation *generated* by directed homotopy with fixed endpoints; composition is given by the concatenation of consecutive paths. The fundamental category of a preordered space can be computed by a van Kampen-type theorem, as proved in [G4], Thm. 3.6, in a much more general setting ('d-spaces', cf. Section 8). The obvious functor $\uparrow\Pi_1(X) \to \Pi_1(UX)$ with values in the fundamental groupoid of the underlying space need neither be full (obviously), nor faithful.

A map $f\colon X \to Y$ induces a functor $f_*\colon \uparrow\Pi_1(X) \to \uparrow\Pi_1(Y)$, a homotopy $\varphi\colon f \to g$ induces a natural transformation $\varphi_*\colon f_* \to g_*$ which generally is *not* invertible. Also because of this, there are crucial differences with the fundamental groupoid $\Pi_1(S)$ of a space, for which a model up to homotopy invariance is given by the skeleton: a family of fundamental groups $\pi_1(S, x_i)$, obtained by choosing one point in each path-connected component of S. For instance, if X is *ordered*, the fundamental category has no isomorphisms nor endomorphisms, except the identities. Thus: (a) the category is *skeletal*; and *ordinary equivalence of categories cannot yield any simpler model*; (b) all the monoids $\uparrow\pi_1(X, x_0) = \uparrow\Pi_1(X)(x_0, x_0)$ are trivial.

Similarly, the singular cubical set $\square X$ consists of all *preorder-preserving* maps $\uparrow\mathbf{I}^n \to X$, and provides the *preordered homology groups* $\uparrow H_n(X)$ (studied in [G6] in a more general setting). All this works because the faces and degeneracies $\uparrow\mathbf{I}^{n-1} \rightleftarrows \uparrow\mathbf{I}^n$ of the ordered cubes *preserve the natural orders*, and could hardly be transferred to tetrahedra.

5. Analysing the fundamental category. An elementary example will give some idea of the analysis developed in [G7] for the fundamental category of a preordered space. (The paper [FRGH], devoted to the analysis of concurrent processes, has similar results, based on different categorical tools, categories of fractions.)

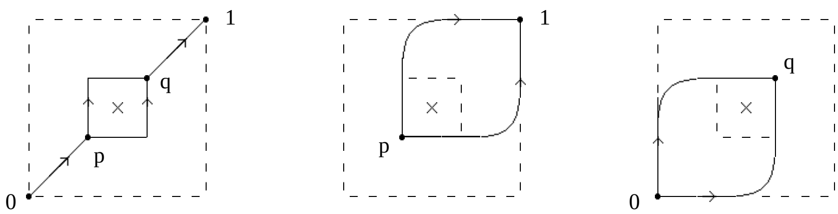
Let us start from the standard *ordered* square $\uparrow[0, 1]^2$, with the euclidean topology and the product order, and consider the (compact) ordered subspace A obtained by taking out the *open* square $]1/3, 2/3[^2$, a sort of 'square annulus'

(1) 

A L L'

Its directed paths, the continuous order-preserving maps $\uparrow[0, 1] \to A$, move 'rightward and upward'. The fundamental category $C = \uparrow\Pi_1(A)$ has *some* arrow $x \to x'$ provided that $x \leq x'$ and both points are in L or L' (the closed subspaces represented above): there are *two* arrows when $x \leq p = (1/3, 1/3)$ and $x' \geq q = (2/3, 2/3)$, and *one* otherwise. This evident fact can be easily proved by the 'van Kampen' theorem recalled above, using the subspaces L, L' (whose fundamental category is the induced order).

Thus, the whole category C is easy to visualise and 'essentially represented' by the full subcategory E on four vertices 0, p, q, 1 (the central cell does not commute)

(2) 

E F P

But E is far from being equivalent to C, as a category, since C *is already a* *skeleton*, in the ordinary sense. In [G7] we have introduced two (dual) directed notions, which take care, respectively, of variation 'in the future' or 'from the past': *future equivalence* (a symmetric version of an adjunction, with two units) and its dual, a *past equivalence* (with two counits); and studied how to extract minimal models for both relations and how to combine them.

In the present case, C has a minimal 'future model' F (the least full reflective subcategory) and a minimal 'past model' P (*co*reflective). The full subcategory E is the join of F and P; it is at the same time *future equivalent* and *past equivalent* to C, and a 'minimal injective model' of C.

Now, the process represented by the ordered space X can be analysed as follows, in the finite model E:

- the action begins at 0, from where we move to the point p,
- p is an (effective) future branching point, where we have to choose between two paths,
- which join at q, an (effective) past branching point,
- from where we can only move to 1.

6. Two-dimensional analysis. In [G8], we have extended this analysis introducing a *strict* fundamental 2-category $\uparrow\Pi_2(X)$. But this construction is complicated, perhaps non natural, and it is not clear whether it can be extended to higher dimension.

More naturally, one can define a fundamental *biased d-lax 2-category* $\uparrow b\Pi_2(X)$, as studied in a work in progress [G9]. It is interesting to note that the geometric guideline gives precise directions for the comparison cells, different from the ones previously considered, by Burroni [Br] and Leinster [Le] for lax 2-categories (biased and unbiased, respectively). The term 'd-lax' refers to this choice, while 'biased' (resp. 'unbiased') refers to structures based on binary (resp. multiple) operations.

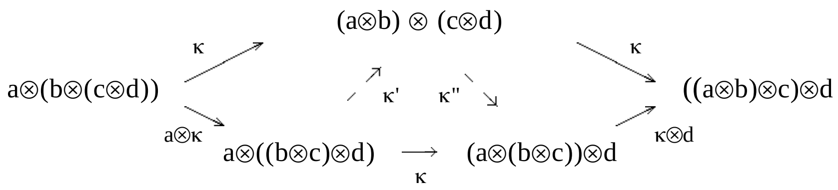
An object of $\uparrow b\Pi_2(X)$ is a point of X, an arrow $a: x \to x'$ is a path, and a cell $[\alpha]: a \to a': x \to x'$ is a homotopy class of homotopies of paths. More precisely, α is a 2-*homotopy* $a \to a'$ (with fixed endpoints), and its class $[\alpha]$ is up to the equivalence relation generated by 3-*homotopies* $\alpha' \to \alpha''$ (with fixed boundary).

Now, writing $a \otimes b$ the concatenation of two consecutive paths, our structure has comparison cells, for units and associativity, directed as follows:

(1) $\lambda: 1_x \otimes a \to a$, $\qquad\rho: a \to a \otimes 1_{x'}$, $\qquad\kappa: a \otimes (b \otimes c) \to (a \otimes b) \otimes c$,

always going from a first concatenation to a second concatenation of the same paths which, at each instant, *has made a longer way than the initial one*.

Then, the coherence theorem for such a structure says that all diagrams (naturally) constructed with comparison cells commute. This remains true in an *extended* structure, where we add *higher* associativity comparisons κ', κ'' depending on four consecutive arrows, which break Mac Lane's pentagon into 3 commutative triangles:

(2) 

A further exension provides an *unbiased* version, $\uparrow u\Pi_2(X)$, with n-ary concatenations $a_1 \otimes ... \otimes a_n$ of consecutive paths.

7. Links with categorical models for biological systems. Unexpectedly again, the analysis of a category through *minimal past and future models*, as sketched above (Section 5) and developed in [G7], has appeared to be closely related with notions recently introduced by A.C. Ehresmann [Eh], within a series of papers with J.P. Vanbremeersch, dedicated to modelling biosystems, neural systems, etc. Likely, because of the common design of studying non-reversible actions.

It would be difficult to fully explain this here. Let us only remark two pairs of neighbouring notions, using the terminology of both papers. First, a *past retract* P of a category X (i.e. a full coreflective subcategory), used in [G7] as a 'past model' of X, is plainly a particular case of a *corefract* (i.e. a full weakly coreflective subcategory) as defined in [Eh] 1.2. Second, one can prove that the *past spectrum* P of a category X *having no* O^-*-branchings*, in the sense of [G7], is necessarily a *root* of X, as defined in [Eh], Section 2. In [G7], many examples of Section 9 fall in this situation: their past spectrum is a root and their future spectrum a coroot.

8. Richer topological settings. In a preordered space, every loop lives in a zone where the preorder is chaotic, *and is reversible*; therefore, this setting has no 'directed circle' or 'directed torus'.

We briefly recall more complex directed structures, which allow for non-reversible loops. All of them contain the directed interval $\uparrow\mathbf{I}$ with the structure considered above, so that all the previous constructions can be easily extended.

(a) In a setting studied in [G4], a *d-space* $X = (X, dX)$ is a topological space equipped with a set dX of (continuous) maps $a: \mathbf{I} \to X$; these maps, called *directed paths* or *d-paths*, must contain all constant paths and be closed under concatenation and (weakly) increasing reparametrisation. A *d-map* $f: X \to Y$ (or *map* of d-spaces) is a continuous mapping between d-spaces which preserves the directed paths: if $a \in dX$, then $fa \in dY$.

The category of d-spaces is written as d**Top**. It has all limits and colimits, constructed as in **Top** and equipped with the initial or final d-structure for the structural maps. Again, the forgetful functor U: d**Top** → **Top** has a left and a right adjoint; a topological space is viewed as a d-space by its *natural* structure, where all (continuous) paths are directed (via the right adjoint to U). Also p**Top** has an obvious functor with values in d**Top**.

Reversing d-paths, by the involution $r(t) = 1 - t$, yields the *reflected*, or *opposite*, d-space $RX = X^{op}$. The *standard directed circle* $\uparrow\mathbf{S}^1 = \uparrow\mathbf{I}/\partial\mathbf{I}$ has the obvious d-structure, where paths have to follow a precise orientation.

(b) Another setting for Directed Algebraic Topology comes from a directed version of Dana Scott's equilogical spaces [Sc, BBS], which was introduced in [G6].

An *inequilogical space* $X = (X^{\#}, \sim_X)$ is a *preordered* topological space $X^{\#}$ endowed with an equivalence relation $\sim_X$ (or $\sim$). The quotient $|X| = X^{\#}/\sim$ is viewed as a preordered topological space (with the induced preorder and topology), or a topological space, or a set, as convenient. A *map* $f: X \to Y$ 'is' a mapping $f: |X| \to |Y|$ which admits some *continuous preorder-preserving* lifting $f': X^{\#} \to Y^{\#}$.

This category is denoted as p**Eql**. The category p**Top** fully embeds in the latter, identifying a preordered space X with the pair $(X, =_X)$. This category has all limits and colimits, and is Cartesian closed (like the one of equilogical spaces). Directed homotopy is defined by the standard directed interval $\uparrow\mathbf{I}$. Various models for the directed circle are considered in [G6]; the simplest is perhaps $(\uparrow\mathbf{R}, \equiv_{\mathbf{Z}})$, i.e. the quotient *in* p**Eql** of the *directed* real line modulo the action of the group of integers.

(c) Finally, let us observe that extending preordered spaces by some *local* notion of ordering, as frequently done in the theory of concurrency, seems not to provide a good setting, with all limits and colimits; the usual attempts cannot realise the cone on the directed circle (cf. [G4], 4.6).

9. Sketching a formal setting for DAT. The phenomena we want to study make sense in a category **A** equipped with 2-cells (homotopies) which, generally, cannot be reversed **-** but *reflected*.

More precisely, as a variation of Kan's notion of a category equipped with an abstract cylinder endofunctor [Ka], **A** is a *dI1-category*. By this we mean that it comes equipped with:

(a) a *reflection* $R: \mathbf{A} \to \mathbf{A}$, i.e. an involutive (covariant) automorphism (also written $R(X) = X^{op}$, $R(f) = f^{op}$),

(b) a *cylinder* endofunctor $I: \mathbf{A} \to \mathbf{A}$, with four natural transformations: two *faces* (∂^α), a *degeneracy* (e) and a *reflection* (r)

(1) $\partial^\alpha: 1 \rightleftarrows I : e, \qquad r: IR \to RI \qquad (\alpha = \pm)$,

satisfying the equations

(2) $e\partial^\alpha = 1: \mathrm{id} \to \mathrm{id}, \qquad RrR.r = 1: IR \to IR,$
$Re.r = eR: IR \to R, \qquad r.\partial^- R = R\partial^+: R \to RI.$

Since $RR = 1$, the transformation r is invertible with $r^{-1} = RrR: RI \to IR$ and

$r.\partial^+R = R\partial^-$.

A *homotopy* $\varphi: f^- \to f^+: X \to Y$ is defined as a map $\varphi: IX \to Y$ with $\varphi.\partial^\alpha X = f^\alpha$ (also written $\hat{\varphi}$ to distinguish it from the homotopy). Each map $f: X \to Y$ has a *trivial* endohomotopy, $1_f: f \to f$, represented by $f.eX = eY.If: IX \to Y$.

Every homotopy $\varphi: f \to g: X \to Y$ has a *reflected homotopy*

(3) $\varphi^{op}: g^{op} \to f^{op}: X^{op} \to Y^{op}$, $\quad (\varphi^{op})\hat{} = R(\hat{\varphi}).r: IRX \to RIX \to RY$,

and $(\varphi^{op})^{op} = \varphi$, $(0_f)^{op} = 0_{(f^{op})}$.

An object X is said to be *reversible* if it coincides with X^{op}, and *reflexive* or *self-dual* if it is isomorphic to the latter. (The structure itself is *reversible* when $R = id\mathbf{A}$; then, a homotopy has a *reversed* homotopy $\varphi^{op}: g \to f$. Such structures have no 'privileged directions'.)

Dually, a *dP1-category* has a reflection (as above) and a *cocylinder*, or *path endofunctor* $P: \mathbf{A} \to \mathbf{A}$, with natural transformations in the opposite direction

(4) $e: 1 \leftrightarrows P: \partial^\alpha$, $\qquad r: RP \to PR$.

satisfying the dual equations.

It is easy to see that a dI1-structure where the cylinder functor has a right adjoint, $I \dashv P$, automatically produces the natural transformations of the cocylinder, and a dP1-structure; we say then that $\mathbf{A}$ is a *dIP1-category*. Both endofunctors produce the same homotopies, represented equivalently by maps $IX \to Y$ or $X \to PY$.

A dI1- or dP1-structure is often generated by a standard directed interval, by cartesian (or tensor) product and internal hom, respectively. In these frames, one can define directed homology and 'first order' homotopy theory. *Much structure has to be added to develop higher directed homotopy theory*, which will be done in a sequel (following the line of [G1, G2] and, for the reversible case, [G3]).

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