# CATEGORY THEORY AND LIVING SYSTEMS *

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**1. The problem.** The phenomenology of living systems (i.e., their relation to the physical world) does not yet have a satisfactory theory. This is not just a philosophical issue. In order to emulate the skill and subtlety of biological computation in our own machines and systems, we need to understand what are the means by which such veridicality is obtained.

Any person is capable of amazing feats of computational accuracy for which current biological accounts are completely inadequate. To awake at a set hour without external clues, to catch a ball hit to some distant point in space (running on dead-reckoning to the exact spot), to execute complex choreography ending exactly with the music (even when that music has only been heard a few times), to measure out in a glance exactly one kilo of coffee beans, these aspects of *exquisite* control seem to require explanation that is principled and coherent.

I believe that mathematical principles play a role in all biological computation, and that a formal categorical substrate underlies the messy ad hoc networks of living systems. Whether or not such a platonic (or really Leibnizian) view is found to hold in nature, it could be helpful for our emulations.

In the remainder of this paper, we sketch some arguments and history for the use of categories in biology in section 2. The origin of weak adjointness (in algebraic topology) is discussed in section 3, while in section 4, the precise definition is given. Finally, the last section considers applications to our basic problem.

**2. Why categories?.** The primal attraction of categories is that they allow all different mathematical viewpoints to be represented. Also, category theory is "vertically" organized (as with organic tissue). For example, one has the hierarchy: object, morphism, category, functor, natural transformation, adjointness, monad, higher-order categories, ..., while in, e.g., functional analysis one has only points, sequences, norms, and spaces. Moreover, categorical entities are subject to a constant process of enrichment, which bears a certain resemblance to evolution.

Such categorical points-of-view have been proposed by Robert Rosen [28], [29] and by René Thom [31]. This is similar to categorical approaches to systems theory (e.g., [1]) and is compatible with the views of Whitehead and Bertalanffy of events as a complex network of interactions (like diagrams in categories).

An independent approach to theoretical biology via category theory was proposed by the author in [17] and [18] which is based on the idea that perception and action can be viewed as adjoint functors.

Our original proposal of coherent mechanisms for action and perception by biological entities aimed at accounting for such phenomena as visual Gestalts and synergies of neuromuscular coordination. In the present work, we consider a generalized concept where the (weak) right adjoint functor is allowed to be multivalued on morphisms and the bijection is only a surjection. We also require that the composition of action followed by perception is the identity.

The concept of adjointness has been applied to situations arising in systems theory, logic, and language by several people. The model of left/right adjoint functors for syntax vs. semantics goes back to Lawvere's seminal paper [22]. Minimal realization

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vs. behavior as a left/right adjoint pair was considered by Goguen [11], [12] and is cited in Mac Lane [24, p. 87]. More recently, the weak adjointness considered below was applied to typed graphs by Banach and Coradini [6]. Arbib and Manes [1] have an interesting interpretation of adjointness in terms of the reversal of time.

For additional work on applications of category theory to systems and biology, see also [10], [4], [13], [20], [21], and [25]. Ehresmann and Vanbremeersch [8], in a series of papers, have substantially enlarged the categorical framework with a detailed and interesting model of how temporal difference play a key role in biological systems; see, e.g., [9].

Zeeman [33] used homology as a rough model of visual perception. In [14] the author generalized adjointness so that homology has a "weak" right adjoint.

**3. Origin in algebraic topology.** To solve problems regarding generalized homology theory and the realization of commutative diagrams [15], [16], we introduced weak adjointness [14]. As an example, for fixed positive integer $n$ the Eilenberg-Mac Lane space $K(G, n)$ which is the space with a single non-vanishing homotopy group $G$ in dimension $n$ [30, p. 424] is a weak right adjoint to $n$-dimensional homology.

The universal coefficient theorem of algebraic topology states that there is a natural short exact sequence, ending in an epimorphism from $H^n(X; G)$, the $n$-th cohomology group with coefficients in $G$, to $\hom(H_n(X), G)$ [30, p. 243]. But there is a natural equivalence of bifunctors [30, p. 428]

$$[X, K(G, n)] \xrightarrow{\cong} H^n(X; G)$$

Putting them together one obtains a natural epimorphism

$$[X, K(G, n)] \to \hom(H_n(X), G)$$

which sends any homotopy class $[f]$ to $f_*$, the induced map in homology. This is an example of a weak adjointness which fits Mac Lane's "right adjoint, right inverse" situation [24, p. 129]; the composition $H_n \circ K(-, n)$ is the identity on abelian groups.

**4. Weak adjointness.** The following development is from Kainen [14] where the notion of weak right adjoint was defined. A related notion appears in Maranda [26]. Seely also uses the term "weak adjointness" [27] but not in a related sense. Our notion of weak adjointness is to ordinary adjointness as a weak initial object is to an ordinary initial object - there is a factorization but it need not be unique.

Let $\mathcal{B}, \mathcal{C}$ be a pair of small categories. Given a functor $N : \mathcal{C} \to \mathcal{B}$ and a function $X$ from $Ob(\mathcal{B}) \to Ob(\mathcal{C})$, there is a *weak right adjunction* $\phi$ from $X$ to $N$ if for all object pairs $C, B$, there are surjections

$$\phi_{C,B} : \mathcal{C}(C, XB) \to \mathcal{B}(NC, B)$$

which are natural in the sense that for every $\gamma : C' \to C$, the following diagram commutes (where $h^\#$ and $h_\#$, resp., mean precomposition and postcompositon by $h$).

$$\begin{array}{ccc} \mathcal{C}(C, XB) & \xrightarrow{\phi_{C,B}} & \mathcal{B}(NC, B) \\ \Big\downarrow \gamma^{\#} & & \Big\downarrow (N\gamma)^{\#} \\ \mathcal{C}(C', XB) & \xrightarrow{\phi_{C',B}} & \mathcal{B}(NC', B) \end{array}$$

A quasifunctor [14] $G : \mathcal{B} \to \mathcal{C}$ is a pair of functions $G_0, G_1$ with $G_0 : Ob(\mathcal{B}) \to Ob(\mathcal{C})$ and $G_1$ is a function from morphisms in $\mathcal{B}$ to nonempty subsets of morphisms in $\mathcal{C}$ between the corresponding objects such that the two obvious properties hold with respect to composition. Namely, if $k : B'' \to B'$ and $c : B' \to B$, then for all $\gamma \in G_1(c)$ and all $\kappa \in G_1(k)$, $\gamma\kappa \in G_1(ck)$ (the composition property) and $1_{G_0(C)} \in G_1(1_C)$.

If there is a weak right adjoint from an object function $G_0$ to a functor $F$, then there is a unique maximal extension of $G_0$ to a quasi-functor. For every $\beta : B \to B'$, let $G_1(\beta)$ consist of the family of all morphisms $\gamma : G_0(B) \to G_0(B')$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}(C, G_0(B)) & \xrightarrow{\phi_{C,B}} & \mathcal{B}(FC, B) \\ \Big\downarrow \gamma_{\#} & & \Big\downarrow \beta_{\#} \\ \mathcal{C}(C, G_0(B')) & \xrightarrow{\phi_{C,B'}} & \mathcal{B}(FC, B') \end{array}$$

Then $G_1$ is the required extension of $G_0$ to a quasi-functor. This idea is used in Mac Lane as well [24, p. 81, Thm 2 (ii)], but there the quasi-functor is an actual functor since, for ordinary adjointness, there is a unique arrow which makes the square commute. But in the case of weak adjointness, as noted in [26], [14], a natural epimorphism is sufficient to guarantee a nonempty set of morphisms and, hence, to define a quasi-functor.

**5. Application to perception and action.** Let $\mathcal{B}$ and $\mathcal{C}$ denote small categories, and assume we are given a functor $N : \mathcal{C} \to \mathcal{B}$. Suppose $\mathcal{C}$ has coproducts and $N$ preserves them. Moreover, let us further assume that the following "solution set" condition holds: For each $B$ in $Ob(\mathcal{B})$ there exists $Sol(B) \subset Ob(\mathcal{C})$ such that if $\beta : NC \to B$, then $\beta = \gamma \circ N(\alpha)$, where $\alpha : C \to C'$ for some $C' \in Sol(B)$ and $\gamma : NC' \to B$. Then by Theorem 2.1 of [14] there is a weak right adjoint $X$ to $N$.

We shall assume that these conditions do hold, thinking of $N$ as the endogenous functor (corresponding to perception) and $X$ as exogenous (corresponding to action). Since $X$ is only a quasi-functor, it is multi-valued on morphisms. We interpret this as a non-uniqueness for biological action.

Finally, we also assume that $NX = 1$, which implies $N$ is full and $X$ is faithful.

A *Gestalt* is the holistic perception of a complex visual relationship as in the sudden realization that a low-resolution black-and-white image represents a Dalmatian dog in dappled shade. Dual to this is the notion of *synergy* (Bernstein [5]) which refers to a complex neuromuscular act involving the simultaneous modulation of muscle tension and joint-angles in order to carry out a physiological task - as in reaching for and grasping a glass of water or signing one's name.

If $N$ is perception, then one may regard the preservation of colimits as the anticipation of consequences - cf. [8], [7]. We proposed that Gestalts amount to the formation of colimits [17] and, in [18], that the formation of synergies is a limit process.

In [14], we showed that if $N$ preserves weak colimits, then the natural epimorphism $\phi_{C,B} : \mathcal{C}(C, XB) \to \mathcal{B}(NC, B)$ induces an epimorphism

$$\phi_{\delta,B} : [\delta, XB] \to [N\delta, B],$$

where $[,]$ denotes the family of natural transformations from one diagram to another, both on the same underlying scheme, where in this case the second diagram is the constant diagram, provided that the diagram $\delta$ has a weak colimit. We interpret this as showing that for any such physical diagram $\delta$ and any biological "goal" $B$, the diagram is related to the corresponding action $XB$ in such a way that the perceived diagram relates to the goal.

If the diagram is suitably restricted (e.g., to a family of arrows with a common target), then a dual result holds: Given any diagram $\delta$ in $\mathcal{B}$, a physical object $C$, and a natural transformation from the constant diagram $NC$ to $\delta$, there is a natural transformation from $C$ to any of the realizations of $X\delta$ (i.e., any selection of morphisms from the nonempty sets to form a diagram in $\mathcal{C}$) which corresponds to it under $\phi$. That is, there is an epimorphism

$$\phi_{C,\delta} : [C, X\delta] \to [NC, \delta],$$

which we interpret as the realization of intent by action.

Lawvere [23] and Tierney [32] have considered the application of categories to physics, where they have used the notion of topos. Using enriched categories and extending strict adjointness from a physical context, we hope for a synthesis with their approach in future work.

In addition, some interesting questions arise in regard to diagram realization when the categories are groupoids (i.e., every morphism is invertible); see [19]. This would correspond to a substrate (quantum?) where the elementary operations are invertible. Non-uniqueness of the weak adjoints could then arise through the failure of coherence in the quantum computation.

REFERENCES


[1] Arbib, M. A. & Manes, E. G., Time-varying systems, in **Category Theory for Compuation and Control** (Proc. of the 1st International Symp., U. Mass., Amherst, 1974, E. G. Manes, Ed.), 1974, pp. 106–112.

[2] Baez, J. & Dolan, J., 2001, **From Finite Sets to Feynman Diagrams: Mathematics Unlimited – 2001 and Beyond**, Berlin: Springer, 29-50.

[3] Baianu, I. C., Computer Models and Automata Theory in Biology and Medicine, *Mathematical modeling* **7** (1986) 1513–1537 (also at cogprints.org).

[4] Baianu, I. C. & Scripcariu, D. On adjoint dynamical systems, *Bull. Math. Biology* **35** (1973) 475–486.

[5] N. Bernstein, **The Coordination and Regulation of Movements** Pergamon Press: Oxford, 1967.

[6] Banach, R. & Corradini, A., Abstract diagrams and an opfibration account of typed graph transformation, 1999, www.cs.man.ac.uk/ banach/some.pubs/Abs.Diag.Op.Ty.Gr.Tr.pdf

[7] Dubois Daniel M., Computing Anticipatory Systems with Incursion and Hyperincursion, in **Computing Anticipatory Systems: CASYS - First International Conference**, American Institute of Physics, AIP Conference Proceedings 437, pp. 3-29, 1998.

[8] Ehresmann, A.C. & Vanbremeersch J.-P., Hierarchical evolutive systems, *Bull. Math. Biol.* **49** (1987) 13–50.

[9] Ehresmann, A.C. & Vanbremeersch J.-P, http://perso.wanadoo.fr/vbm-ehr/AnintroT.htm

[10] H. Ehrig & H.-J. Kreowski, Power and initial automata in pseudoclosed categories, in **Category Theory for Compuation and Control** (Proc. of the 1st International Symp., U. Mass., Amherst, 1974, E. G. Manes, Ed.), 1974, pp. 162–169.

[11] Goguen, J. A., Systems and minimal realization, **IEEE Conf. on Decision and Control**, Miami Beach, 1971.

[12] Goguen, J. A., Semantics of computation, in **Category Theory for Compuation and Control** (Proc. of the 1st International Symp., U. Mass., Amherst, 1974, E. G. Manes, Ed.), 1974, pp. 234–240.

[13] Harnad, S., The symbol grounding problem. *Physica D*, **42** (1990) 335–346.

[14] Kainen, P. C., Weak adjoint functors, *Math. Zeit.*, **122** (1971) 1–9.

[15] Kainen, P. C., Universal coefficient theorems for generalized homology and stable cohomotopy, *Pacific Math. J.* **37** (1971) 397–407.

[16] Kainen, P. C., Every commutative square of abelian groups can be realized by a homotopy commutative square of Moore or co-Moore spaces, *J. London Math. Soc.* **5** (1972) 577–583.

[17] Kainen, P.C., Functorial cybernetics of attention, Chap. 57 in **Neurocomputers and Attention**, vol. II, A. V. Holden and V. I. Kryukov, Eds., Manchester University Press, 1990.

[18] Kainen, P. C., Categorical coordination of muscular action. In **Proc. 8th International Conference on Cybernetics and Systems**, C. N. Manikopoulos, Ed., Vol. I, NJIT Press, Newark, NJ, 1990, pp.314–319.

[19] Kainen, P. C., Isolated squares in hypercubes and robustness of commutativity, *Cahiers de topol. et géom. diff. cat.*, **XLIII** (2002) 213–220.

[20] Kelso, J. A. S. & Clark, J. E., Eds., **The Development of movement control and coordination** Imprint Chichester [West Sussex]; New York: Wiley, 1982.

[21] Kelso, J. A. S. **Dynamic patterns : the self-organization of brain and behavior** Imprint Cambridge, Mass.: MIT Press, 1995.

[22] Lawvere, F. W., Functorial semantics of algebraic theories, *Proc. Nat. Acad. of the USA* **50** (1963) 870–872.

[23] Lawvere, F. W., Categorical Algebra for Continuum Micro Physics, *Journal of Pure and Applied Algebra* **175** no. 1-3 (2002) 267–287.

[24] Mac Lane, S., **Categories for the Working Mathematician**, New York: Springer, 1971.

[25] Maia, T. V., & Chang, N. C., Grounding the acquisition of grammar in sensorimotor representations, AAAI Spring Symposium on Learning Grounded Representations, Stanford, CA, 2001.

[26] Maranda, J. M., Injective structures, *Trans. American Math. Soc.* **110** (1964) 98–135.

[27] Seely, R. A. G., Weak adjointness in proof theory, in **Applications of Sheaves** (Proc. Research Symp. on Appl. of Sheaf Theory to Logic, Algebra, and Analysis, Durham, 1977), Berlin: Springer, 1979, pp. 697–701.

[28] Rosen, R., A relational theory of biological systems. *Bulletin of Mathematical Biophysics* **20** (1958) 245–260.

[29] Rosen, R., Relational biology and the origin of life. In K. Matsuno et al., **Molecular Evolution and Protobiology**, New York: Plenum Press, pp. 421431, 1984.

[30] Spanier, E. H., **Algebraic Topology**, New York: McGraw-Hill, 1966.

[31] Thom, R., **Semio-physics**, Reading, MA: Addison-Wesley, 1990.

[32] Tierney, M., Sheaf Theory and the Continuum Hypothesis, in **Toposes, Algebraic Geometry and Logic**, F.W. Lawvere, Ed., Springer Lecture Notes in Mathematics **274**, (1972), 13–42.

[33] Zeeman, E. C., The topology of the brain and visual perception, in **Topology of 3-manifolds and related topics** (Proc. The Univ. of Georgia Institute, 1961), pp. 240–256, Prentice-Hall, 1962.